IGA Lecture III: Twisted Spin$_c$ structures

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Review: Spin_c-structures

- (V, B) a finite-dimensional Euclidean vector space,
- $\mathbb{C}\mathrm{l}(V)$ complex Clifford algebra: generators $v \in V$, relations

$$vv' + v'v = 2B(v, v').$$

Then $\mathbb{C}\mathrm{l}(V)$ is a finite-dimensional C^*-algebra.

Similarly, for a finite rank Euclidean vector bundle $V \to X$ with fiber metric B define a complex Clifford bundle $\mathbb{C}\mathrm{l}(V) \to X$.

Let $V \to X$ be a Euclidean vector bundle, $\mathrm{rank}(V)$ even.

Definition

A Spin_c-*structure* on V is a $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathsf{S} \to X$ with a $*$-isomorphism

$$\varrho\colon \mathbb{C}\mathrm{l}(V) \to \mathrm{End}(\mathsf{S}).$$

S is called the spinor module.

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Remarks

- The definition is equivalent to an orientation on V together with a lift of the structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}_c(n)$. (Connes, Plymen.)
- If V has odd rank, one defines a Spin_c-structure on V to be a Spin_c-structure on $V \oplus \mathbb{R}$.

Let $V \to X$ be a Euclidean vector bundle.

Example

Suppose $J \in \Gamma(\mathrm{O}(V))$ is a complex structure, $J^2 = -\,\mathrm{id}_V$. Get $V^{\mathbb{C}} = V^+ \oplus V^-$. Then

$$\mathsf{S} = \wedge(V^+)$$

defines a Spin_c-structure on V, with $\varrho(v) = \sqrt{2}(\epsilon(v^+) + \iota(v^-))$ for $v \in V$.

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Example

Suppose $\omega \in \Gamma(\wedge^2 V^*)$ is symplectic; let R_ω be the corresponding skew-adjoint endomorphism. Then

$$J_\omega = \frac{R_\omega}{|R_\omega|} \in \Gamma(\mathrm{O}(V))$$

is a complex structure, defining a Spin_c-structure on V.

Spin$_c$-structures

Basic properties

- Any two Spin$_c$-structure S, S' on V differ by a line bundle:

$$\mathsf{S}' = \mathsf{S} \otimes L \leftrightarrow L = \mathrm{Hom}_{\mathbb{C}\mathrm{l}}(\mathsf{S}, \mathsf{S}').$$

- Obstructions to existence of Spin$_c$-structure:

$$W_3(V) \in H^3(X, \mathbb{Z}), \quad w_1(V) \in H^1(X, \mathbb{Z}_2).$$

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Example

The dual S^* of a spinor module is again a spinor module. Get a line bundle

$$K_{\mathsf{S}} = \mathrm{Hom}_{\mathbb{C}\mathrm{l}}(\mathsf{S}, \mathsf{S}^*)$$

called the canonical line bundle for S. Note

$$K_{\mathsf{S}\otimes L} = K_{\mathsf{S}} \otimes L^{-2}.$$

Spin$_c$-structures

If M is a manifold with a smooth Spin$_c$-structure S, one defines the Spin$_c$-Dirac operator

$$\partial\!\!\!/ \colon \Gamma(\mathsf{S}) \xrightarrow{\nabla} \Gamma(TM \otimes \mathsf{S}) \xrightarrow{\varrho} \Gamma(\mathsf{S}).$$

If $L \to M$ is a line bundle, denote by $\partial\!\!\!/_L$ the Spin$_c$-Dirac operator for $\mathsf{S} \otimes L$.

Quantization of Hamiltonian G-spaces (in a nutshell)

Hamiltonian G-space $\Phi\colon M \to \mathfrak{g}^*$

1. $\iota(\xi_M)\omega = -\mathrm{d}\langle \Phi, \xi \rangle$,
2. $\mathrm{d}\omega = 0$,
3. $\ker(\omega) = 0$.

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For q-Hamiltonian spaces already Step 1 fails, since ω may be degenerate.

Review: q-Hamiltonian G-spaces

Let G be a compact Lie group, and $\cdot$ an invariant inner product on $\mathfrak{g} = \mathrm{Lie}(G)$.

Definition

A q-Hamiltonian G-space (M, ω, Φ) is a G-manifold M, with $\omega \in \Omega^2(M)^G$ and $\Phi \in C^\infty(M, G)^G$, satisfying

1. $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi$,
2. $\mathrm{d}\omega = -\Phi^*\eta$,
3. $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$.

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Problems:

- There is no notion of 'compatible almost complex structure'
- In general, q-Hamiltonian G-spaces need not even admit Spin_c-structures.

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Example

- $G = \mathrm{Spin}(5)$ has a conjugacy class $\mathcal{C} \cong S^4$ (does not admit almost complex structure).
- $G = \mathrm{Spin}(2k+1),\ k > 2$ has a conjugacy class not admitting a Spin_c-structure.

However, we will show that q-Hamiltonian spaces carry '*twisted*' Spin_c-structures.

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The definition of the twisted Spin_c-structures involves *Dixmier-Douady bundles*

Dixmier-Douady theory

Notation:

- H separable complex Hilbert space, possibly $\dim H < \infty$,
- $\mathbb{B}(H)$ bounded linear operators,
- $\mathbb{K}(H)$ compact operators $(= \overline{\mathbb{B}_{\mathrm{fin}}(H)})$

Fact: $\mathrm{Aut}(\mathbb{K}(H)) = \mathrm{PU}(H)$ (strong topology).

Definition

A DD-bundle $\mathcal{A} \to X$ is a $\mathbb{Z}_2$-graded bundle of $*$-algebras modeled on $\mathbb{K}(H)$, (for H a $\mathbb{Z}_2$-graded Hilbert space), with structure group the even part of $\mathrm{PU}(H)$.

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Theorem (Dixmier-Douady)

The obstruction to writing $\mathcal{A} = \mathbb{K}(\mathcal{E})$, with $\mathcal{E}$ a $\mathbb{Z}_2$-graded bundle of Hilbert spaces, is a class

$$\mathrm{DD}(\mathcal{A}) \in H^3(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}_2).$$

Hence, the trivially graded DD bundles give a 'realization' of $H^3(X, \mathbb{Z})$, similar to line bundles 'realizing' $H^2(X, \mathbb{Z})$.

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Remark

This framework is not convenient for a Chern-Weil theory. A more differential-geometric realization is given by the theory of bundle gerbes.

Definition

Let $\mathcal{A}_1 \to X_1,\ \mathcal{A}_2 \to X_2$ be *DD*-bundles. A Morita morphism

$$(\Phi, \mathcal{E})\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$$

is a map $\Phi\colon X_1 \to X_2$ together with a $\mathbb{Z}_2$-graded bundle $\mathcal{E} \to X_1$ of bimodules

$$\Phi^*\mathcal{A}_2 \circlearrowright \mathcal{E} \circlearrowleft \mathcal{A}_1,$$

locally modeled on $\mathbb{K}(H_2) \circlearrowright \mathbb{K}(H_1, H_2) \circlearrowleft \mathbb{K}(H_1)$.

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Remark

- $(\Phi, \mathcal{E})\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ *exists if and only if* $\mathrm{DD}(\mathcal{A}_1) = \Phi^* \, \mathrm{DD}(\mathcal{A}_2)$.
- *Any two Morita bimodules* $\mathcal{E}, \mathcal{E}'$ *differ by a line bundle:*

$$\mathcal{E}' = \mathcal{E} \otimes L \leftrightarrow L = \mathrm{Hom}_{\Phi^*\mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}, \mathcal{E}').$$

Example

$V \to X$ Euclidean vector bundle of even rank $\Rightarrow \mathbb{C}\mathrm{l}(V)$ is a DD-bundle. A Morita trivialization

$$(p, \mathsf{S}^{\mathrm{op}})\colon \mathbb{C}\mathrm{l}(V) \dashrightarrow \mathbb{C}$$

is a Spin_c-structure. The DD-class is given by

$$\mathrm{DD}(\mathsf{S}) = (W^3(V), w_1(V)) \in H^3(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}_2).$$

From Dirac structures to DD bundles

Review of linear Dirac structures

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- For $\Theta\colon V_1 \to V_2$ and $\omega \in \wedge^2 V_1^*$ write

$$v_1 + \mu_1 \sim_{(\Theta,\omega)} v_2 + \mu_2 \Leftrightarrow \begin{cases} v_2 = \Theta(v_1) \\ \mu_1 = \Theta^*(\mu_2) + \omega(v_1, \cdot) \end{cases}$$

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- It defines a Dirac morphism $(\Theta, \omega) \colon (\mathbb{V}_1, E_1) \dashrightarrow (\mathbb{V}_2, E_2)$ if every element of E_2 is related to a unique element of E_1.

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- The definitions extend to vector bundles $V \to X$.

From Dirac structures to DD bundles

Example

- Hamiltonian G-spaces are described as G-equivariant Dirac morphisms

$$(\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow (\mathcal{T}\mathfrak{g}^*, E_{\mathfrak{g}^*}).$$

- q-Hamiltonian G-spaces are described as G-equivariant Dirac morphisms

$$(\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}G_\eta, E_G).$$

- There is a multiplication morphism

$$(\mathrm{Mult}_G, \varsigma)\colon (\mathbb{T}G_\eta, E_G) \times (\mathbb{T}G_\eta, E_G) \dashrightarrow (\mathbb{T}G_\eta, E_G).$$

The Dirac-Dixmier-Douady functor

Theorem (Alekseev-M, 2010)

There is a functor from Dirac structures on vector bundles $V \to X$ to DD-bundles:

$$E \mapsto \mathcal{A}_E.$$

Furthermore, there are canonical Morita isomorphisms

$$\mathbb{C}\mathrm{l}(V) \dashrightarrow \mathcal{A}_V, \quad \mathbb{C} \dashrightarrow \mathcal{A}_{V^*}$$

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Furthermore, there are canonical Morita isomorphisms

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N.B.: We identify two Morita morphisms $\mathcal{E}, \mathcal{E}' \colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ if they are related by a trivial line bundle.

Example

The Cartan Dirac structure $(\mathbb{T}G_\eta, E_G)$ defines a DD-bundle $\mathcal{A}^{\mathsf{Spin}} := \mathcal{A}_{E_G} \to G$. The 'multiplication morphism' for the Cartan Dirac structure gives a morphism

$$\mathrm{Mult}_*\colon \mathcal{A}^{\mathsf{Spin}} \times \mathcal{A}^{\mathsf{Spin}} \dashrightarrow \mathcal{A}^{\mathsf{Spin}}.$$

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Example

A q-Hamiltonian G-space (M, ω, Φ) defines a Dirac morphism

$$(\mathrm{d}\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}G_\eta, E_G).$$

Hence we get a Morita morphism

$$\mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathcal{A}_{TM} \dashrightarrow \mathcal{A}_{E_G} = \mathcal{A}^{\mathsf{Spin}},$$

a 'twisted Spin_c-structure'.

Construction of the DDD functor $E \mapsto \mathcal{A}_E$

Outline

1. From $E \subset \mathbb{V}$, construct family of skew-adjoint operators $D_x,\ x \in X$ acting on real Hilbert spaces $\mathcal{V}_x$.
2. From $D = \{D_x\}$, construct family of 'polarizations' of $\mathcal{V}_x$.
3. From the polarization, construct DD-bundle $\mathcal{A} \to X$.

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Inspired by and/or similar to:
Carey-Mickelsson-Murray 1997, Lott 2002, Atiyah-Segal 2004, Freed-Hopkins-Teleman 2005, Bouwknegt-Mathai-Wu 2011.

Step 1: Constructing $\{D_x,\ x \in X\}$

Assume $X = \mathrm{pt}$, so V is a vector space.

Choice of Euclidean metric B identifies

$$\mathrm{Lag}(\mathbb{V}) \cong \mathrm{O}(V).$$

Here $A \in \mathrm{O}(V)$ corresponds to

$$E = \{((A - I)v, \tfrac{1}{2}(A + I)v) \in \mathbb{V} = V \oplus V^* |\ v \in V\}.$$

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Define skew-adjoint operator $D_E = \frac{\partial}{\partial t}$ on $\mathcal{V} = L^2([0,1], V)$, with domain

$$\mathrm{dom}(D_E) = \{f\colon\ f(1) = -Af(0)\}.$$

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$E = V^*$ corresponds to $A = I$, and $f(1) = -Af(0)$ are anti-periodic boundary conditions. Note $\ker(D_E) = 0$.

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Note that in general, $\ker(D_E) = \ker(A + I) = E \cap V$.

Thus, if $V \to X$ is a vector bundle, the choice of a Euclidean metric takes us from Dirac structures $(\mathbb{V}, E)$ to skew-adjoint Fredholm families

$$D_E = \{(D_E)_x,\ x \in X\},$$

where $(D_E)_x$ is $\frac{\partial}{\partial t}$ on $\mathcal{V}_x = L^2([0,1], V_x)$, with boundary conditions determined by E_x.

Step 2: Polarizations

Let $\mathcal{V}$ be a real Hilbert space. Recall $A \in \mathbb{B}(\mathcal{V})$ is Hilbert-Schmidt if $\mathrm{tr}(A^*A) < \infty$.

Definition

An even polarization on $\mathcal{V}$ is an equivalence class of orthogonal complex structures $J \in \mathrm{O}(\mathcal{V})$, where

$$J \sim J' \Leftrightarrow J - J' \text{ is Hilbert-Schmidt.}$$

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An odd polarization on $\mathcal{V}$ is an even polarization on $\mathcal{V} \oplus \mathbb{R}$.

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Lemma

The even polarization defined by $J = \frac{D+S}{|D+S|}$ does not depend on choice of S.

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Lemma

The even polarization defined by $J = \frac{D+S}{|D+S|}$ does not depend on choice of S.

If $\dim\ker(D)$ odd, replace $\mathcal{V}$ with $\mathcal{V} \oplus \mathbb{R}$, and obtain odd polarization.

Step 3: The Dixmier-Douady bundle

- $\mathcal{V}$ a real Hilbert space.
- $\mathbb{C}\mathrm{l}(\mathcal{V})$ its complex Clifford algebra.
- $\mathsf{S}_J = \overline{\wedge \mathcal{V}_+}$ spinor module defined by $J \in \mathrm{O}(\mathcal{V}),\ J^2 = -\mathrm{id}_{\mathcal{V}}$ (Hilbert space completion).

Theorem (Shale-Stinespring, 1965)

For orthogonal complex structures J, J' on $\mathcal{V}$,

$$\dim \mathrm{Hom}_{\mathbb{C}\mathrm{l}}(\mathsf{S}_J, \mathsf{S}_{J'}) = \begin{cases} 1 & \text{if } J \sim J' \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{K}(\mathsf{S}_J) = \mathbb{K}(\mathsf{S}_{J'})$ canonically if $J \sim J'$.

Step 3: The Dixmier-Douady bundle

From $(\mathbb{V}, E)$ we constructed the family D_E of skew-adjoint Fredholm operators on $\mathcal{V} = \bigcup_{x\in X},\ \mathcal{V}_x = L^2([0,1], V)$, which in turn defines a polarization on $\mathcal{V}$.

Use fiberwise representatives J_x to define

$$\mathcal{A}_x = \mathbb{K}(\mathsf{S}_{J_x}).$$

Then $\mathcal{A} = \bigcup_x \mathcal{A}_x$ is a well-defined DD-bundle.

Remark

- $\ker(D_E) \cong E \cap V$.
- *Hence if* $E = V^*$, *then* $\ker(D_E) = 0$, *and* $\mathcal{A} = \mathbb{K}(\mathsf{S}_J)$ *for* $J = \frac{D_E}{|D_E|}$.
- *If* $E = V$, *then* $\ker(D_E) = V$, *and* $\mathcal{V} = V \oplus V^\perp$. *This explains* $\mathbb{C}\mathrm{l}(V) \dashrightarrow \mathcal{A}$.

Example

For the Cartan-Dirac structure $\mathbb{T}G, E)$, get family

$$D_g = \frac{\partial}{\partial t}, \quad \mathrm{dom}(D_g) = \{f \in L^2([0,1],\mathfrak{g}) |\ f(1) = -\operatorname{Ad}_g f(0)\}.$$

Let $\mathcal{A}^{\mathrm{Spin}} := \mathcal{A}_{E_G}$. If G is connected, then

$$\mathrm{DD}(\mathcal{A}^{\mathrm{Spin}}) \in H^3(G,\mathbb{Z}) \times H^1(G,\mathbb{Z}_2)$$

is the pull-back of the generators of $H^3(\mathrm{SO}(\mathfrak{g}),\mathbb{Z}) = \mathbb{Z}$ resp. $H^1(\mathrm{SO}(\mathfrak{g}),\mathbb{Z}_2) = \mathbb{Z}_2$ under $\operatorname{Ad}\colon G \to \mathrm{SO}(\mathfrak{g})$. (See Atiyah-Segal.)

In particular, if G simple and simply connected, then

$$\mathrm{DD}(\mathcal{A}^{\mathrm{Spin}}) = \mathsf{h}^\vee x$$

where $x \in H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ is the generator, and $\mathsf{h}^\vee$ is the dual Coxeter number.

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Corollary

Suppose (M, ω, Φ) is a q-Hamiltonian G-space. Then

$$W_3(M) = \mathsf{h}^\vee \Phi^* x, \;\; w_1(M) = 0.$$

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Corollary

Suppose (M, ω, Φ) is a q-Hamiltonian G-space. Then

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This follows from existence of $\mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathcal{A}^{\mathrm{Spin}}$.
In particular, this result applies to the conjugacy classes of G.