A Computational Program in Non-Commutative Geometry

Emil Prodan

Yeshiva University, New York, USA

Topological matter, strings, K-theory, and related areas

Work supported by U.S. NSF grant CAREER-DMR-1056168 and by Keck Foundation

Part 1: Homogeneous Disordered Crystals

Homogeneous Lattice Systems

Let:

1. $\mathcal{H} = \mathbb{C}^N \otimes \ell(\mathbb{Z}^d)$ be the physical Hilbert space
2. $H \in \mathbb{B}(\mathcal{H})$ a Hamiltonian describing an infinit sample

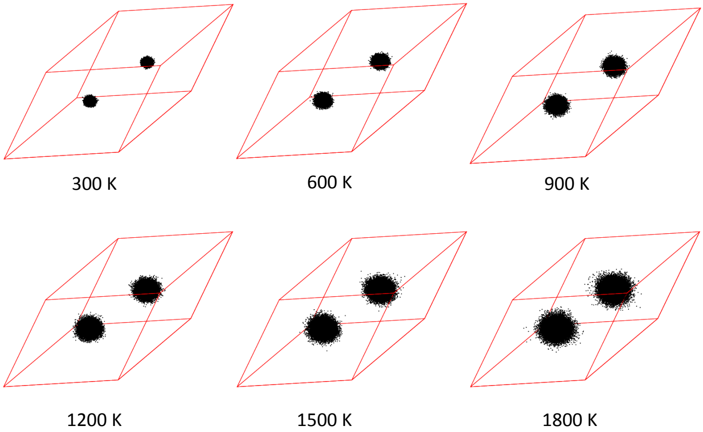
Definition (Bellissard, 1986)

Consider the set of all translates of H:

$$\Omega = \{ T_a H T_a^* \, , \, a \in \mathbb{Z}^d \} \subset \mathbb{B}(\mathcal{H}) \, .$$

The condensed matter system described by H is homogeneous if Ω has a compact closure in the strong topology of $\mathbb{B}(\mathcal{H})$.

Example: Thermally disordered crystal



Quantum Molecular Dynamics of 1000 atoms Si crystal simulated by Thomas Kühne.

2400 K

2700 K

3000 K

Quantum Molecular Dynamics of 1000 atoms Si crystal simulated by Thomas Kühne.

1. The configurations are encoded as (for smaller temperatures):

$$\omega = \{R_x^\alpha\}_{x\in\mathbb{Z}^d}^{\alpha=\overline{1,N_a}} \in \Omega = \prod_{x\in\mathbb{Z}^d}\prod_{\alpha=1}^{N_a}\Omega_0^\alpha = \prod_{x\in\mathbb{Z}^d}\Omega_0 \ ,$$

($N_a = \#$ of atoms in the primitive cell)

2. The translations act by shifting the configuration of atoms:

$$\mathbb{Z}^d \ni q \ \to \ \tau_y\omega = \tau_y\{R_x^\alpha\} = \{R_{x-y}^\alpha\}.$$

3. Gibbs probability measure:

$$\mathrm{d}\mathbb{P}(\omega) = \mathcal{Z}_V^{-1} e^{-\beta\mathcal{V}_V(\omega)} \prod_{x\in V\subset\mathbb{Z}^d}\prod_{\alpha=1}^{N_a}\mathrm{d}\omega_x^\alpha\Big|_{V\to\infty} \ ,$$

$\mathcal{V}_V$ = atomic potential, $\beta = \frac{1}{kT}$.

Conjecture: (D. Ruelle, 1968)

For a pure crystalline phase, the Gibbs measure is invariant and ergodic relative to the translations.

Conclusion:

A homogeneous crystalline phase is defined by a measure preserving ergodic dynamical system:

$$(\Omega, \mathbb{Z}^d, \tau, \mathrm{d}\mathbb{P})$$

and the dynamics of the electrons by a covariant family:

$$\{H_\omega\}_{\omega \in \Omega} \; , \quad T_a H_\omega T_a^* = H_{\tau_a \omega} \; .$$

Proposition:

The bounded covariant Hamiltonians on $\mathbb{C}^d \otimes \ell^2(\mathbb{Z}^d)$ take the following form:

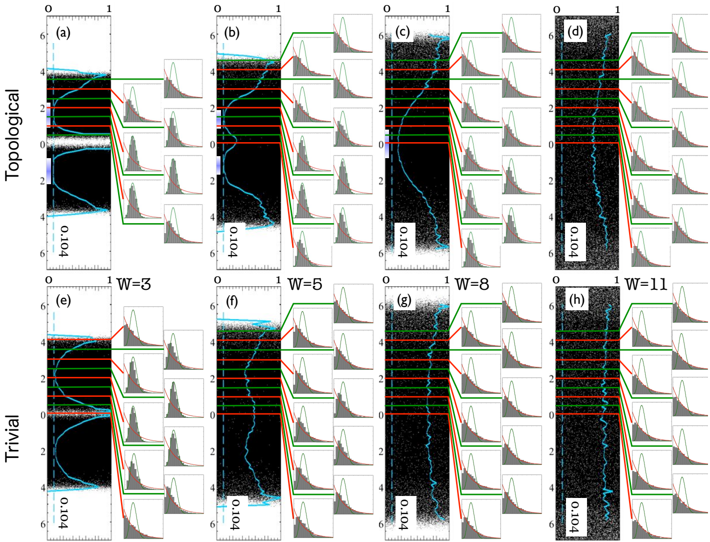
$$H_\omega = \sum_{q\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} w_q(\tau_x\omega) \otimes |x\rangle\langle x|\, T_q$$

When uniform magnetic fields are present, then the ordinary translations T_q are replaced by the magnetic translations.

The spectrum of H_ω can be:

1. Anderson localized $\Rightarrow$ direct transport coefficients vanish at $T = 0$
2. Extended $\Rightarrow$ finite direct transport coefficients at $T = 0$

Example: Spectra of Homogeneous Hamiltonians



Part 2: Classification of Homogeneous Crystals

Classification of Homogeneous Crystalline Systems

A. P. Schnyder, S. Ryu, A. Furusaki, A. W. W. Ludwig, *Classification of topological insulators and superconductors in three spatial dimensions*, Phys. Rev. **B 78**, 195125 (2008).
A. Kitaev, *Periodic table for topological insulators and superconductors*, (Advances in Theoretical Physics: Landau Memorial Conference) AIP Conference Proceedings **1134**, 22-30 (2009).
S. Ryu, A. P. Schnyder, A. Furusaki, A. W. W. Ludwig, *Topological insulators and superconductors: tenfold way and dimensional hierarchy*, New J. Phys. **12**, 065010 (2010).

j	TRS	PHS	CHS	CAZ	$0,8$	1	2	3	4	5	6	7
0	0	0	0	A	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
1	0	0	1	AIII		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
0	$+1$	0	0	AI	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$+1$	$+1$	1	BDI	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
2	0	$+1$	0	D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
3	-1	$+1$	1	DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
4	-1	0	0	AII	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
5	-1	-1	1	CII		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
6	0	-1	0	C			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
7	$+1$	-1	1	CI				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- each $n \in \mathbb{Z}$ or $\mathbb{Z}_2$ defines a distinct macroscopic insulating phase: $\sigma_{xx} = 0$.
- the phases are separated by a bulk Anderson transition: $\sigma_{xx} > 0$
- $\sigma_{\|} > 0$ along any boundary cut into the crystals.

The Index Theorem for Bulk Projections ($d =$ even)

Let d be even and let P_ω be a covariant projection such that:

$$\int_\Omega d\mathbb{P}(\omega)\ \Big\langle 0 \Big| \big|[X, P_\omega]\big|^d \Big| 0 \Big\rangle < \infty$$

Let $\Gamma_1, \ldots, \Gamma_2$ be irreducible rep of $\mathcal{C}l_d$. Then, $\mathbb{P}$-almost surely

$$F_\omega = P_\omega \left(\frac{X \cdot \Gamma}{|X|} \right)_{+-} P_\omega \in \text{Fredholm class}$$

and

$$\boxed{\mathrm{Ind}\, F_\omega = \Lambda_d \sum_{\rho \in S_d} (-1)^\rho \int_\Omega d\mathbb{P}_\omega \Big\langle 0 \Big| P_\omega \prod_{i=1}^{d} \imath \big[X_{\rho_i}, P_\omega\big] \Big| 0 \Big\rangle}$$

J. Bellissard, A. van Elst, H. Schulz-Baldes, *The non-commutative geometry of the quantum Hall effect*, J. Math. Phys. **35**, 5373-5451 (1994).

E. P., B. Leung, J. Bellissard, *The non-commutative n-th Chern number* ($n \geq 1$), J. Phys. A: Math. Theor. **46**, 485202 (2013).

The Index Theorem for Bulk Unitaries ($d = $ odd)

Let d be odd and let U_ω be a covariant unitary such that:

$$\int_\Omega d\mathbb{P}(\omega)\ \Big\langle 0 \Big| \big|[X, U_\omega]\big|^d \Big| 0 \Big\rangle < \infty$$

Let E_+ be the spectral projection onto the positive spectrum of $X \cdot \Gamma$. Then, $\mathbb{P}$-almost surely

$$F_\omega = E_+ U_\omega E_+ \in \text{Fredholm class}$$

and

$$\boxed{\mathrm{Ind}\, F_\omega = \Lambda_d \sum_{\rho \in S_d} (-1)^\rho \int_\Omega d\mathbb{P}(\omega) \Big\langle 0 \Big| \prod_{i=1}^{d} \imath U_\omega^* \big[X_{\rho_i}, U_\omega\big] \Big| 0 \Big\rangle}$$

E. P. and H. Schulz-Baldes, *Non-commutative odd Chern numbers and topological phases of disordered chiral systems*, J. Funct. Anal. **271**, 1150-1176 (2016).

Part 3: The algebraic framework of Jean Bellissard

The Algebra of Bulk Physical Observables $\mathcal{A}_d$

Definition (ϕ_{ij} = magnetic flux through the facets of unit cell)

The universal C*-algebra

$$\mathcal{A}_d \;=\; C^*\Big(C_N(\Omega), u_1, \cdots, u_d\Big)\,, \quad C_N(\Omega) = C(\Omega, M_{N\times N})$$

generated by the following commutation relations:

$$\left.\begin{array}{ll} u_i u_i^* = u_i^* u_i = \mathbf{1}, & i = 1, \ldots, d \\ u_i\, u_j \;= e^{i\phi_{ij}} u_j\, u_i, & i, j = 1, \ldots, d \end{array}\right\} non - commutative\ torus$$

$$f\, u_j \;=\; u_j\,(f \circ \tau_j)\,, \qquad \forall\ f \in C_N(\Omega)\,,\ \ j = 1, \ldots, d\,.$$

A generic element takes the form

$$a \;=\; \sum_{x \in \mathbb{Z}^d} a_x\, u_x, \quad a_x \in C_N(\Omega)\,, \quad u_x \;=\; u_1^{x_1} \cdots u_d^{x_d}\;.$$

Canonical Representation on $\mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d)$

Proposition

$$\pi_\omega(u_q) \;=\; T_q\,, \quad q \in \mathbb{Z}^d\,,$$

$$\pi_\omega(f) \;=\; \sum_{x\in\mathbb{Z}^d} f(\tau_x\omega) \otimes |x\rangle\langle x|, \quad \forall\, f \in C_N(\Omega),$$

defines a family $\{\pi_\omega\}_{\omega\in\Omega}$ of $\mathbb{P}$-almost sure faithful representations.

For generic elements

$$\mathcal{A}_d \ni a = \sum_{q\in\mathbb{Z}^d} a_q\, u_q \longrightarrow \pi_\omega(a) = \sum_{x,q\in\mathbb{Z}^d} a_q(\tau_x\omega) \otimes |x\rangle\langle x|\, T_q.$$

All homogeneous lattice models can be generated from $\mathcal{A}_d$!

Fourier Calculus for Algebra of Observables

- Defined by the group of continuous $*$-automorphisms ρ_k induced by the $U(1)$ twists

$$u_j \to e^{\imath k_j} u_j, \qquad k_j \in [0, 2\pi],\ j = 1, \ldots, d.$$

- The Fourier coefficients of $a \in \mathcal{A}_d$

$$\Phi_x(a) = \int_{\mathbb{T}^d} dk\ e^{-\imath \langle x|k\rangle} \rho_k(a u_x^*) \in C_N(\Omega), \quad x \in \mathbb{Z}^d.$$

- For a generic element $a \in \mathcal{A}_d$, the Cesàro sums converge to a

$$a^{(n)} = \sum_{x \in [-n,\ldots,n]^d} \prod_{j=1}^{d} \left(1 - \frac{|x_j|}{n+1}\right) \Phi_x(a)\, u_x$$

Non-Commutative Calculus

Defined over $\mathcal{A}_d$ through the Fourier calculus:

Derivation:

$$\Phi_x(\partial_j a) = -\imath x_j \Phi_x(a), \ j = \overline{1, d}$$

For a generic element

$$a = \sum_{x \in \mathbb{Z}^d} a_x u_x \quad \to \quad \partial_j a = -\imath \sum_{x \in \mathbb{Z}^d} x_j \, a_x \, u_x$$

Integration:

$$\mathcal{T}(a) = \int_\Omega \mathbb{P}(d\omega) \, \mathrm{tr}\big\{\Phi_0(a)\big\}, \quad \mathcal{T}(a) = \int_\Omega d\mathbb{P}(\omega) \, \mathrm{tr}\{a_0(\omega)\}$$

The trace $\mathcal{T}$ over $\mathcal{A}_d$ is continuous, normalized and $\mathcal{T}(\partial_j a) = 0$.

- Chern Numbers (Schulz-Baldes et al, JMP 1994, EP et al 2013, EP et al 2014):

$$C_{even}(p) = \Lambda_d \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T}\big(p \prod_{i=1}^{d} \partial_{\sigma_i} p\big), \quad C_{odd}(u) = \Lambda_d \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T}\big(\prod_{i=1}^{d} u^* \partial_{\sigma_i} u\big)$$

- Finite-Temperature Kubo-formula (Schulz-Baldes & Bellissard in 1990's):

$$\sigma_{ij} = -\mathcal{T}\left((\partial_i h) * (\Gamma + \mathcal{L}_h)^{-1} \partial_j \Phi_{\mathrm{FD}}(h)\right).$$

- Electric polarization (Schulz-Baldes and Teufel in Comm. Math. Phys. 2012):

$$\Delta \mathbf{P} = \int_0^T dt \; \mathcal{T}\left(p(t)[\partial_t p(t), \nabla p(t)]\right)$$

- Orbital magnetization (Schulz-Baldes and Teufel in Comm. Math. Phys. 2012):

$$M_j = \frac{i}{2} \mathcal{T}\left(|h - \epsilon_F| [\partial_{j+1} p, \partial_{j+2} p]\right)$$

- Magneto-Electric Response in $d = 3$ (Leung and EP in J. Phys. A 2013):

$$\Delta \alpha = \tfrac{1}{2} \int dt \; \sum_{\sigma \in S_4} (-1)^\sigma \mathcal{T}\big(p \prod_{i=1}^{4} \partial_{\sigma_i} p\big), \quad \text{(4-th direction = time)}$$

Part 4: The Approximation Program

The smooth sub-algebra

Let

$$C^n(\mathcal{A}_d) = \mathrm{Span}\{a \in \mathcal{A}_d \,|\, \rho_\lambda(a) = n\text{-times differentiable}\}$$

Then

$$\mathcal{A}_d^\infty \;=\; C^\infty(\mathcal{A}_d) \;=\; \bigcap_{n\geq 1} C^n(\mathcal{A}_d) \quad \text{(smooth algebra)}$$

endowed with the topology induced by the semi-norms:

$$\|a\|_\alpha \;=\; \|\partial^\alpha a\| \,, \qquad \partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d} \,, \;\; \alpha \;=\; (\alpha_1, \ldots \alpha_d)$$

Proposition:

$\mathcal{A}_d^\infty$ is a dense Fréchet sub-algebra of $\mathcal{A}_d$, stable under the functional calculus with smooth functions. If $a \in \mathcal{A}_d^\infty$, then its Fourier coefficients decay fast:

$$x^\alpha \|a_x\|_{C_N(\Omega)} \leq \|\partial^\alpha a\| \leq \infty, \quad x^\alpha = x^{\alpha_1}\ldots x^{\alpha_d}.$$

The Sobolev spaces $\mathcal{W}_{r,p}(\mathcal{A}, \mathcal{T}_0)$

Defined as the closure of $\mathcal{A}_d^\infty$ under the norms:

$$\|a\|_{r,p} \;=\; \sum_{x\in\mathbb{Z}^d} \big(1+|x|\big)^r \Big(\int d\mathbb{P}(\omega)|a_x(\omega)|^p\Big)^{\frac{1}{p}} \;, \quad r\in\mathbb{N}\;, \;\; p\in\mathbb{N}_+ \;. \qquad (1)$$

The Sobolev algebra $\bar{\mathcal{A}}_d^\infty$

Defined as the Frechét algebra defined by the closure of $\mathcal{A}_d^\infty$ in the topology defined by the norms $\|\;\|_{r,p}$, $r\in\mathbb{N}$, $p\in\mathbb{N}_+$.

Proposition:

Let $h\in\mathcal{A}_d$ be a self-adjoint element with a mobility gap Δ. Then, $G(h)\in\bar{\mathcal{A}}_d^\infty$ for any Borel function G with support in a mobility gap.

The Periodic Approximating Algebra $\tilde{\mathcal{A}}_d$

Definition:

$$\widetilde{\Omega} = \{\omega \in \Omega \mid \tau_j^{2L+1}\omega = \omega\}, \quad \tilde{\tau}_a = \tau_a|_{\widetilde{\Omega}} \, .$$

Then:

$$\tilde{\mathcal{A}}_d \; = \; C^*\Big(C_N(\widetilde{\Omega}), u_1, \cdots, u_d\Big) \, ,$$

with same commutation relations as before. Let $\tilde{\mathrm{q}} : \Omega \to \widetilde{\Omega}$ be the canonical map and define $d\widetilde{\mathbb{P}} = \tilde{\mathrm{q}}_* \, d\mathbb{P}$. Then the non-commutative manifold $(\tilde{\mathcal{A}}_d, \tilde{\partial}, \widetilde{\mathfrak{T}})$ can be defined as before.

Proposition:

$$\tilde{\mathrm{p}} : \mathcal{A}_d \to \tilde{\mathcal{A}}_d \; , \quad \sum_{x \in \mathbb{Z}^d} a_x u_x \to \sum_{x \in \mathbb{Z}^d} \tilde{a}_x u_x , \quad \tilde{a}_x = a_x|_{\widetilde{\Omega}} \, . \tag{2}$$

is an epimorphism of C^*-algebras.

Finite-Volume Disorder Configurations

$$0 \longrightarrow \big((2L+1)\mathbb{Z}\big)^d \xrightarrow{\;i\;} \mathbb{Z}^d \underset{s}{\overset{\mathrm{ev}}{\rightleftarrows}} \widehat{\mathbb{Z}}^d = \big(\mathbb{Z}/(2L+1)\mathbb{Z}\big)^d \longrightarrow 0\,.$$

1. $\hat{x}$ denotes the class in $\widehat{\mathbb{Z}}^d$ of $x \in \mathbb{Z}^d$
2. Splitting map s is fixed to $s(\hat{x}) = y$, $y \in V_L$ such that $\hat{y} = \hat{x}$.

Definition: The space of disorder configurations at finite volume

$$\widehat{\Omega} = \prod_{\hat{x}\in\widehat{\mathbb{Z}}^d} \Omega_0, \quad \hat{\tau}_y(\hat{\omega}) = \hat{\tau}_y\{\hat{\omega}_{\hat{x}}\}_{\hat{x}\in\widehat{\mathbb{Z}}^d} = \{\hat{\omega}_{\widehat{x-y}}\}_{\hat{x}\in\widehat{\mathbb{Z}}^d}\,.$$

Proposition:

$$\tilde{\omega} = \{\tilde{\omega}_x\}_{x\in\mathbb{Z}^d} \to \hat{\mathrm{q}}\tilde{\omega} \in \widehat{\Omega}, \quad (\hat{\mathrm{q}}\tilde{\omega})_{\hat{x}} = \tilde{\omega}_{s(\hat{x})}\,, \quad d\widehat{\mathbb{P}} = \hat{\mathrm{q}}_*\, \widetilde{d\mathbb{P}}$$

defines an isomorphisms of dynamical systems

$$(\widetilde{\Omega}, \tilde{\tau}, \mathbb{Z}^d, d\widetilde{\mathbb{P}}) \simeq (\widehat{\Omega}, \hat{\tau}, \mathbb{Z}^d, d\widehat{\mathbb{P}})\,.$$

The Finite Approximating Algebra

Definition:

$$\hat{\mathcal{A}}_d \;=\; C^*\Big(C_N(\widehat{\Omega}), \hat{u}_1, \cdots, \hat{u}_d\Big)\ ,$$

with same commutation relations but the additional constraint:

$$(\tilde{u}_j)^{2L+1} = 1, \quad j = 1, \ldots, d.$$

The algebra is well defined only if $\phi_{ij} = \frac{2\pi}{2L+1} \times$ integer, since

$$\big(\hat{u}_i \hat{u}_j \hat{u}_i^*\big)^{2L+1} = e^{\imath(2L+1)\phi_{ij}} (\hat{u}_j)^{2L+1}.$$

Proposition:

$$\hat{\mathrm{p}} : \tilde{\mathcal{A}}_d \to \hat{\mathcal{A}}_d\ , \quad \hat{\mathrm{p}}(u_j) = \hat{u}_j\ , \quad \hat{\mathrm{p}}(\tilde{f}) = \tilde{f} \circ \hat{\mathrm{q}}^{-1}\ ,$$

is a $*$-epimorphism of C^*-algebras.

Approximate Non-Commutative Calculus:

For generic $\hat{a} = \sum_{x\in V_L} \hat{a}_x\, \hat{u}_x$:

$$\hat{\partial}_j \hat{a} = -\imath \sum_{x\in V_L} x_j \hat{a}_x\, \hat{u}_x, \quad \widehat{\mathcal{T}}(\hat{a}) = \int_{\widehat{\Omega}} d\widehat{\mathbb{P}}(\hat{\omega})\, \hat{a}_0(\hat{\omega}).$$

Under the canonical representations:

$$\hat{\pi}_{\hat{\omega}}(\hat{\partial}_j \hat{a}) = \sum_{\lambda^{2L+1}=1} c_\lambda\, \lambda^{\hat{X}} \hat{\pi}_{\hat{\omega}}(\hat{a}) \lambda^{-\hat{X}}\ , \quad c_\lambda = \begin{cases} \frac{\lambda^L}{1-\lambda}, & \lambda \neq 1, \\ 0, & \lambda = 1. \end{cases}$$

$$\widehat{\mathcal{T}}(\hat{a}) = \int_{\widehat{\Omega}} d\widehat{\mathbb{P}}(\hat{\omega})\, \langle 0|\hat{\pi}_{\hat{\omega}}(\hat{a})|0\rangle = \frac{1}{|V_L|} \sum_{x\in V_L} \int_{\widehat{\Omega}} d\widehat{\mathbb{P}}(\tilde{\omega})\, \langle x|\hat{\pi}_{\hat{\omega}}(\hat{a})|x\rangle\ .$$

At this point we found the optimal replacement:

$$\imath[A_\omega, X_j] \to \sum_{\lambda^{2L+1}=1} c_\lambda\, \lambda^{\hat{X}} \hat{A}_{\hat{\omega}} \lambda^{-\hat{X}}, \quad A_\omega = \pi_\omega(a), \quad \hat{A}_{\hat{\omega}} = \hat{\pi}_{\hat{\omega}}\big(\hat{\mathrm{p}} \circ \tilde{\mathrm{p}}(a)\big)$$

This is the foundation for our finite-volume algorithm.

The Approximation Scheme in a Nutshell

Take:

$$L_{n+1} = cL_n + \frac{c}{2} - \frac{1}{2} \to V_{L_{n+1}} = cV_{L_n} \tag{3}$$

Then we have the projective tower of C^*-algebras:

$$C(\widetilde{\Omega}_{L_0}) \overset{\tilde{p}_0}{\twoheadleftarrow} \widetilde{C}(\widetilde{\Omega}_{L_1}) \overset{\tilde{p}_1}{\twoheadleftarrow} \ldots \overset{\tilde{p}_{k-1}}{\twoheadleftarrow} C(\widetilde{\Omega}_{L_n}) \ldots , \quad \varprojlim C(\widetilde{\Omega}_{L_n}) = C(\Omega) \, .$$

and for $\hat{\mathfrak{U}}_d^{(L)} = C^*(\hat{u}_1, \ldots, \hat{u}_d)$:

$$\hat{\mathfrak{U}}_d^{(L_0)} \overset{\hat{p}_0}{\twoheadleftarrow} \hat{\mathfrak{U}}_d^{(L_1)} \overset{\hat{p}_1}{\twoheadleftarrow} \ldots \overset{\hat{p}_{k-1}}{\twoheadleftarrow} \hat{\mathfrak{U}}_d^{(L_n)} \ \ldots , \quad \varprojlim U_d^{(L_n)} = C^*(u_1, \ldots, u_d) \, .$$

Then the approximating scheme is summarized by:

$$\hat{\mathcal{A}}_d^{(L_0)} \overset{\tilde{p}_0}{\twoheadleftarrow} \hat{\mathcal{A}}_d^{(L_1)} \overset{\tilde{p}_1}{\twoheadleftarrow} \ldots \overset{\tilde{p}_{k-1}}{\twoheadleftarrow} \hat{\mathcal{A}}_d^{(L_n)} \ldots , \quad \mathcal{A}_d = \varprojlim \hat{\mathcal{A}}_d$$

(together with the canonical approximate differential calculus)

Convergence to TD-Limit: Smooth Correlations

Assumptions:

a1. The Hamiltonian h belongs to the smooth algebra $\mathcal{A}_d^\infty$.

a2. For any $K \in \mathbb{N}$, the Fourier coefficients of the Hamiltonian satisfy:

$$\|h_q(\omega) - h_q(\omega')\| \leq \frac{A_K}{(1+|V_M|)^K} \ , \quad 0 < A_K < \infty \ , \tag{4}$$

whenever $\omega_x = \omega'_x$ for $x \in V_M$, $M \in \mathbb{N}$.

Theorem:

Let $h \in \mathcal{A}_d$ satisfying a1-a2. Define $\hat{h} \in \hat{A}_d$ as $\hat{h} = (\hat{\mathrm{p}} \circ \tilde{\mathrm{p}})(h)$. Then, for any $K \in \mathbb{N}$, $K \geq 2$, there exists the finite positive constant A_K such that:

$$\Big|\mathcal{T}\big(\partial^{\alpha_1} G_1(h) \ldots \partial^{\alpha_n} G_n(h)\big) - \widehat{\mathcal{T}}\big(\hat{\partial}^{\alpha_1} G_1(\hat{h}) \ldots \hat{\partial}^{\alpha_n} G_n(\hat{h})\big)\Big| \leq \frac{A_K}{(1+|V_L|)^K} \ ,$$

where G_i's are smooth functions on the spectrum of h.

Convergence to TD-Limit: Non-Smooth Correlations

c1. The Hamiltonian $h \in \mathcal{A}_d$ is of finite range and takes the linearized form:

$$h = \sum \big(w_q + \lambda_q \omega_0\big) u_q \,.$$

c2. The Hamiltonian has a mobility gap Δ:

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega) \left\|(h-z)_x^{-1}(\omega)\right\|^s \leq A_s(\delta) e^{-\gamma_s(\delta)|x|} \,, \quad s \in (0,1),\ \delta > 0 \,,$$

uniformly for all $z \in \mathbb{C} \setminus \sigma(h)$ with $\mathrm{dist}(z, \sigma(h) \setminus \Delta) \geq \delta$.

Theorem:

Let $h \in \mathcal{A}_d$ satisfying c1-c2. Define $\hat{h} \in \hat{A}_d$ as $\hat{h} = (\hat{\mathrm{p}} \circ \tilde{\mathrm{p}})(h)$. Then, for any $K \in \mathbb{N}$, there exists the finite positive constant A_K such that:

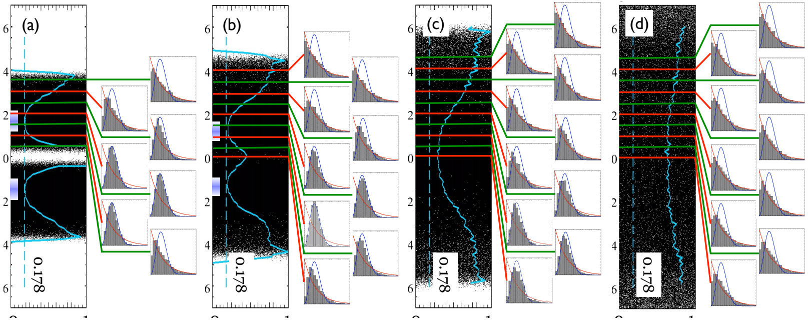
$$\Big|\mathcal{T}\big(\partial^{\alpha_1} G_1(h) \ldots \partial^{\alpha_n} G_n(h)\big) - \widehat{\mathcal{T}}\big(\hat{\partial}^{\alpha_1} G_1(\hat{h}) \ldots \hat{\partial}^{\alpha_n} G_n(\hat{h})\big)\Big| \leq \frac{A_K}{(1+|V_L|)^K} \,,$$

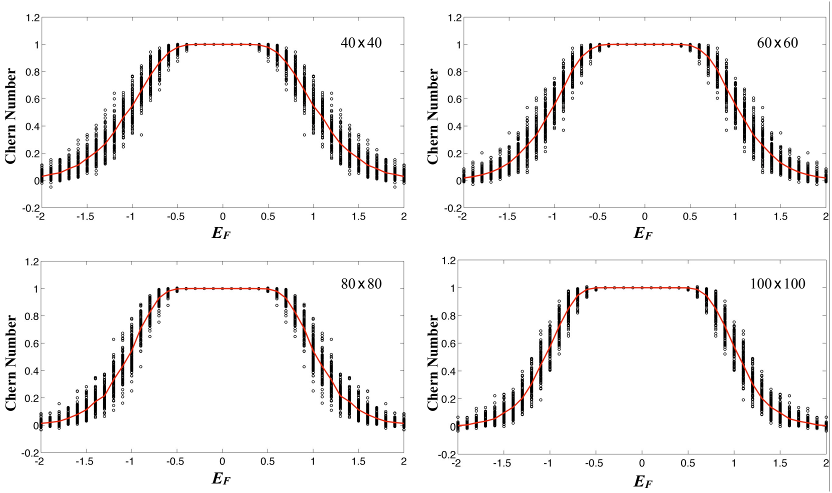
where G_i's are Borel functions that are smooth away from the mobility gap of h.

The Haldane Model (generator of class A in 2D)

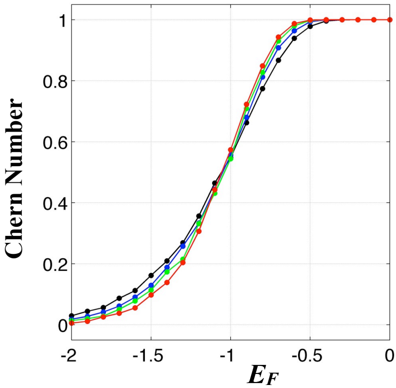
Disordered Haldane model ($\alpha_x = \pm 1$)

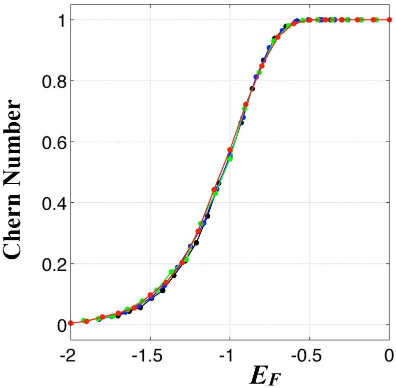
$$H_\omega = \sum_{\langle x,y \rangle} |x\rangle\langle y| + 0.6\imath \sum_{\langle\langle x,y \rangle\rangle} \alpha_x \big(|x\rangle\langle y| - |y\rangle\langle x| \big) + \lambda \sum_x \omega_x |x\rangle\langle x|.$$





Corresponds to the first row - second column case of the previous case. Note the absence of a spectral gap!





The Chern lines overlap almost perfectly after a rescaling of the energy axis

$$E \to E_c + (E - E_c) * (L/L_0)^{\nu} \ , \quad \Lambda \sim \frac{A}{|E - E_c|^{\nu}}$$

($\nu = 2.6$, in line with the most recent estimates)

Table: Numerical values for average Chern numbers

Energy	40 $\times$ 40	60 $\times$ 60	80 $\times$ 80	100 $\times$ 100
-2.0000000000000000	0.0293885304649968	0.0183147848896676	0.0134785966919230	0.0055726403061233
-1.8999999999999999	0.0442301583027775	0.0274502505545331	0.0200229343621875	0.0112501012411246
-1.8000000000000000	0.0563736772645283	0.0416811880195335	0.0285382576963500	0.0259995275657507
-1.7000000000000000	0.0868202901241971	0.0612803850743208	0.0506852078002088	0.0377798251819264
-1.6000000000000001	0.1121154018269069	0.0905166860071905	0.0781754600177580	0.0554182299457663
-1.5000000000000000	0.1617580454580226	0.1291516191502659	0.1133966598848624	0.0977984662347778
-1.3999999999999999	0.2093536896403097	0.1883311262238442	0.1733092018533850	0.1386844139850113
-1.3000000000000000	0.2687556358733589	0.2575144956897765	0.2146703753513447	0.2040079233029510
-1.2000000000000000	0.3565352143319771	0.3333569482253110	0.3319133571108642	0.3066419928551302
-1.1000000000000001	0.4646789224167249	0.4444784219466996	0.4310440221933989	0.4427699238861748
-1.0000000000000000	0.5479958396159215	0.5561471440680733	0.5442615536532044	0.5738596277941682
-0.9000000000000000	0.6624275864985472	0.6798953821199148	0.7086514094234754	0.7228749266484203
-0.8000000000000000	0.7742005453064691	0.8124137607528051	0.8270271100278364	0.8487923693232788
-0.7000000000000000	0.8672349391630054	0.9079791895178040	0.9301639459675241	0.9432234611493278
-0.6000000000000000	0.9392873717233425	0.9636994770114942	0.9802652381114992	0.9872940741308633
-0.5000000000000000	0.9784417158133359	0.9935074963179980	0.9974987656403326	0.9988846769813913
-0.4000000000000000	0.9958865415757685	0.9992024708366942	0.9998527876642247	0.9999656328302596
-0.3000000000000000	0.9998184404341747	0.9999824660477071	0.9999988087144891	0.9999996457562911
-0.2000000000000000	0.9999952917010211	0.9999977443008894	0.9999999997655000	0.9999999999862120
-0.1000000000000000	0.9999999046002306	0.9999999998972079	0.9999999999998473	0.9999999999999849
0.0000000000000000	0.9999999963422543	0.9999999999988873	0.9999999999999996	0.9999999999999999

AIII Model:

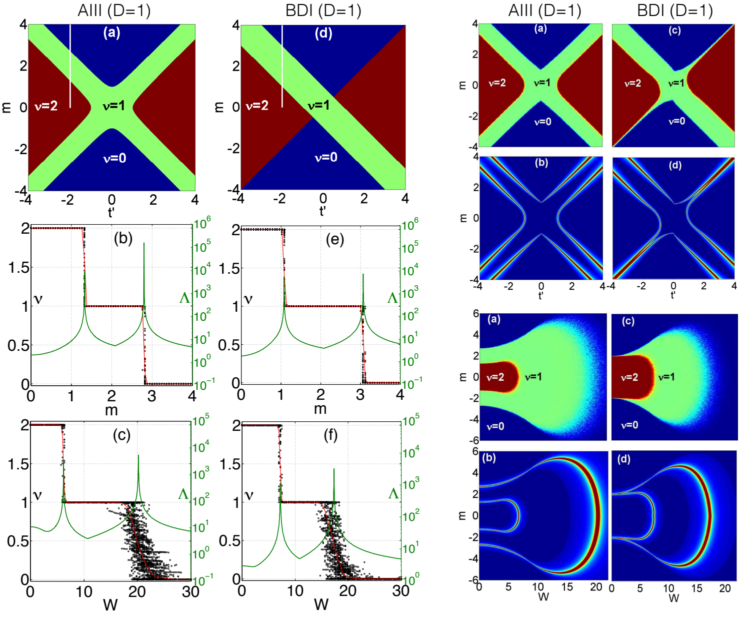
$$
\begin{aligned}
(H\psi)_x = m_x\ \hat{\sigma}_2\psi_x \\
&+ \frac{1}{2}\ t_x[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+1} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-1}] \\
&+ \frac{1}{2}\ t'[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+2} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-2}],
\end{aligned}
$$

BDI Model:

$$
\begin{aligned}
(H\psi)_x = m_x\ \hat{\sigma}_1\psi_x \\
&+ \frac{1}{2}\ t_x[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+1} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-1}] \\
&+ \frac{1}{2}\ t'[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+2} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-2}].
\end{aligned}
$$

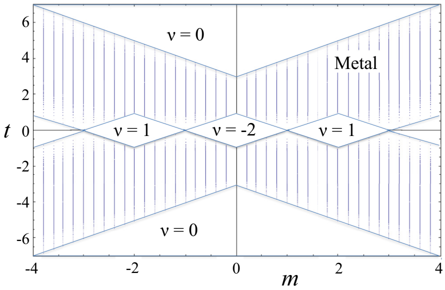
The disorder is present in the first-neighbor hopping and in the onsite potential:

$$
t_x = t + W_1\omega_x, \quad m_x = m + W_2\omega'_x, \quad \omega_x, \omega'_x \in [-\tfrac{1}{2}, \tfrac{1}{2}].
$$


AIII (D=1)
BDI (D=1)
(a)
(b)
(c)
(d)
(e)
(f)
ν=2
ν=1
ν=0
m
t'
W
ν
Λ

The Model: (Γ's generate Cl_5)

$$(H\psi)_x = \frac{1}{2}\sum_{j=1}^{3}\left\{\imath\Gamma_j(\psi_{x-e_j}-\psi_{x+e_j})+\Gamma_4(\psi_{x-e_j}+\psi_{x+e_j})\right\}$$
$$+\,\imath t\Gamma_1\Gamma_3\Gamma_4+(m+W\omega_x)\Gamma_4\psi_x$$



Phase diagram at $W = 0$.

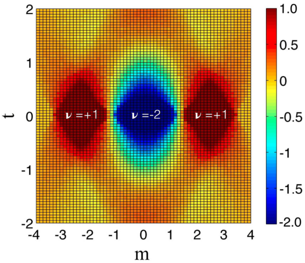


FIG. 3. (Color online) The phase diagrams in the phase space (m,t) at disorder strength $W=4$. The computations were completed on a cubic lattice of $N=16\times16\times16$ unit cells, following the procedure described in the text.

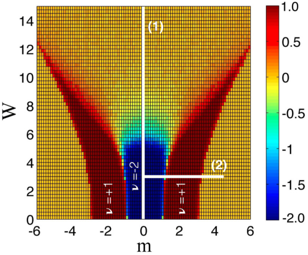


FIG. 4. (Color online) The phase diagrams in the phase space (m,W) at $t=0$. The computations for ν were done with a cubic lattice of $N=16\times16\times16$ unit cells.

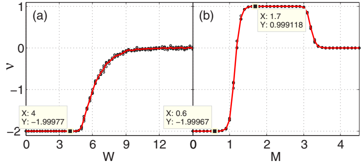


FIG. 5. (Color online) Evolution of the winding number ν with disorder W (a) and parameter m (b). The raw, unaveraged data for five disorder configurations are shown by the scattered points and the