Subfactors and Modular Tensor Categories

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Outline

- Motivation
- What is a modular tensor category?
- Where do modular tensor categories come from?
- Some examples of modular tensor categories coming from subfactors.
- Conjectures and questions
Modular Tensor Categories (MTC’s) are algebraic structures that arise in a number of related areas of mathematics and mathematical physics.

- Representations of conformal field theories (e.g. rational vertex operator algebras) form MTC’s. (Moore-Seiberg, Huang)

- Witten and Reshitikhin-Turaev constructed invariants of links and 3-manifolds and Topogical Quantum Field Theories (TQFT) from MTC’s.

- Freedman, Kitaev, Wang have developed a model for quantum computation based on TQFT/MTC - Microsoft Station Q.
What looks like a group but is not a group?

A naive but often fruitful idea in mathematics is to generalize mathematical objects by listing their essential properties as axioms, and then dropping one or more of them.

- By dropping the parallel postulate from Euclid’s geometry, one obtains interesting non-Euclidean geometries, such as hyperbolic geometry.

- If one describes the properties of an algebra of functions on a (nice) topological space and drops the requirement that multiplication of functions is commutative, one obtains the notion of a C*-algebra.

- More generally, Connes’ theory of noncommutative geometry studies noncommutative analogues of algebras of functions on geometric spaces.
Let $G$ be a finite group, and consider $\text{Rep}_G$, the category of representations of $G$ on finite-dimensional (complex) vector spaces.

Here the objects of $\text{Rep}_G$ are representations, and the morphisms are linear maps which intertwine the action of $G$.

$\text{Rep}_G$ is a $\mathbb{C}$-linear Abelian category with finitely many simple objects, and by Maschke’s Theorem it is semisimple.
Other properties of $\text{Rep}_G$:

- **Monoidal** - can take $\otimes$; natural isomorphisms
  \[ \alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z \]
  satisfying certain coherence relations (associativity constraint); trivial representation is tensor identity.

- **Rigid + Spherical** - objects have duals with evaluation and coevaluation maps
  \[ e_X : X \otimes X^* \to 1, \quad c_X : 1 \to X^* \otimes X \]
  satisfying “zig-zag” relations; natural isomorphism $X \to X^{**}$ with left traces equal to right traces.

- **Symmetric** - natural isomorphisms
  \[ b_{X,Y} : X \otimes Y \to Y \otimes X \]
  satisfying the hexagon relations (“braiding”), such that
  \[ b_{Y,X} \circ b_{X,Y} = ld_{X \otimes Y}. \]

- **(Unitary** - monoidally equivalent to category of unitary reps, which has $\mathbb{C}^*$ structure)
A fusion category is a $\mathbb{C}$-linear semisimple rigid monoidal category with finitely many simple objects and simple identity object.

We have seen that for a finite group $G$, $\text{Rep}_G$ is a fusion category.

Conversely, every symmetric fusion category is equivalent to $\text{Rep}_G$ for some finite (super)group $G$. (Deligne)

What about non-symmetric fusion categories?
Definition

A **braided** fusion category is a fusion category with natural isomorphisms

\[ b_{X,Y} : X \otimes Y \to Y \otimes X \]

satisfying the *hexagon relations*. (Joyal-Street)

How symmetric is a braiding?

Definition

The **Mueger center** of a braided fusion category \( C \) is

\[ \{ X \in C : b_{X,Y} \circ b_{Y,X} = id_{Y \otimes X}, \ \forall Y \} \]

A braided fusion category is symmetric iff it is equal to its Mueger center.

Definition

A **modular tensor category** is a braided spherical fusion category with trivial Mueger center.
Modular tensor categories have an amazingly intricate structure, first discovered by Moore and Seiberg in the context of conformal field theory.

For a MTC $\mathcal{C}$, one can define $S$ and $T$ matrices as follows. The entries of $S$ are given by the (normalized) values of Hopf links labeled by pairs of simple objects of $\mathcal{C}$. $T$ is a diagonal matrix with entries given by twists of the simple objects.

Then $S$ and $T$ are unitary matrices, and the map

$$
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \rightarrow S, 
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \rightarrow T
$$

defines a projective representation of the modular group $SL_2(\mathbb{Z})$.

The fusion coefficients $N^Z_{X \otimes Y}$ of $\mathcal{C}$ are determined from the $S$-matrix by the Verlinde formula

$$
N^Z_{X \otimes Y} = \sum_r \frac{S_{X_i,X_r} S_{X_j,X_r} S_{X_k,X_r}}{S_{1,X_r}}.
$$
Where do modular tensor categories come from?

1. Categories of representations of quantum groups with parameter a root of unity give MTC’s (via a semisimplification procedure).

2. Let \( \mathcal{C} \) is a monoidal category. The Drinfeld center \( Z(\mathcal{C}) \) is the monoidal category of half-braidings of \( \mathcal{C} \) by objects \( X \) of \( \mathcal{C} \). (Objects of \( Z(\mathcal{C}) \) are pairs \((X, b_X)\), where \( b_X \) is a natural isomorphism \( X \otimes Y \to Y \otimes X \) satisfying the braiding relation).

If \( \mathcal{C} \) is a spherical fusion category, then \( Z(\mathcal{C}) \) is a MTC.

All known examples of MTCs come from these two constructions.
Where do (not-necessarily-modular) fusion categories come from?

1. Representations of (quantum) groups
2. Subfactors

- A **von Neumann algebra** is a *-algebra which is the commutant of a unitary group representation.

- A **factor** is a von Neumann algebra with trivial center.

- A **subfactor** is a unital inclusion $N \subseteq M$ of ($\infty$-dim, finite-trace) factors.
Let $N \subseteq M$ be a subfactor.

The **index** $[M : N]$ of a subfactor is the Murray-von Neumann coupling constant of $N L^2(M)$.

**Theorem (Jones ’83)**

$[M : N] \in \{4\cos^2 \frac{\pi}{k}\}_{k=3,4,5\ldots} \cup [4, \infty]$.

Given a finite index subfactor $N \subseteq M$, one can consider the category $\mathcal{N}$ of $N - N$ bimodules $\otimes$-generated by $N M N$.

- $\mathcal{N}$ is a semisimple rigid monoidal category, but **not** necessarily finite.

- If $\mathcal{N}$ is finite, it is a fusion category.

- **Every** unitary fusion category can be realized this way as a subcategory of such an $\mathcal{N}$.
Subfactors with index \( < 4 \) \( \iff \) fusion categories associated to quantum \( SU(2) \).

In the 1990’s, 2 “exotic” subfactors were discovered with index slightly above 4

1. Haagerup subfactor (index \( \frac{5 + \sqrt{13}}{2} \))

2. Asaeda-Haagerup subfactor (index \( \frac{5 + \sqrt{17}}{2} \))

More recently, the extended Haagerup subfactor was constructed by Bigelow-Morrison-Peters-Snyder (non-quadratic index).
Let \( \mathcal{C} \) be a fusion category. Let \( \text{Inv}(\mathcal{C}) \) be the tensor subcategory of invertible objects.

On a decategorified level, \( \text{Inv}(\mathcal{C}) \) gives a finite group \( G \), which acts on the set of \( (\cong\)-classes of) simple objects of \( \mathcal{C} \).

**Definition**

A fusion category \( \mathcal{C} \) is **pointed** if \( \mathcal{C} = \text{Inv}(\mathcal{C}) \).

Every pointed fusion category is equivalent to \( \text{Vec}_G^\omega \), the category of \( G \)-graded vector spaces, with associator given by \( \omega \in H^3(G, \mathbb{C}^*) \).
Definition

A fusion category $C$ is **quadratic** if there are exactly two orbits of simple objects under the action of $\text{Inv}(C)$.

**Ex:** The fusion category associated to the Haagerup subfactor has simple objects

\[ \{g\}_{g \in G} \cup \{gX\}_{g \in G} \]

with fusion rules

\[ g \cdot h = gh, \quad gX = Xg^{-1}, \quad X^2 = 1 + \sum_{g \in G} gX. \]

where $G = \mathbb{Z}_3$.

A category with fusion rules as above for some finite Abelian group $G$ is called an Izumi-Haagerup category.
In the 1990’s, Izumi introduced a general method for constructing quadratic fusion categories from endomorphisms of $C^*$ algebras.

- Very explicit description of the categories - allows one to perform computations in the Drinfeld center.
- Explicit polynomial equations whose solutions give Izumi-Haagerup categories for groups of odd order.
- Solved the equations for $\mathbb{Z}_3$ and $\mathbb{Z}_5$ and found the modular data of the Drinfeld centers.
More recently, Evans and Gannon solved Izumi’s equations for many more cyclic groups of odd order and gave a general formula for modular data associated to Izumi-Haagerup subfactors for cyclic group of odd order.

This modular data appears to be a “graft” of modular data associated to dihedral groups and to certain quantum groups.

Based on this Evans-Gannon argued that the Haagerup subfactor should not be viewed as “exotic” at all.

However there is still no general construction for Izumi-Haagerup categories, and it is not known whether Evans-Gannon’s infinite series of modular data is realized by a corresponding series of MTC’s.
It turns out the Asaeda-Haagerup subfactor is also related to an Izumi-Haagerup category, but in a more complicated way: its fusion category is Morita equivalent to a $\mathbb{Z}_2$ orbifold of an I-H category for the group $\mathbb{Z}_4 \times \mathbb{Z}_2$ (G-Izumi-Snyder).

The modular data of the Asaeda-Haagerup subfactor has a similar structure to the Evans-Gannon modular data, and can be generalized to a series for the groups $\mathbb{Z}_{4n} \times \mathbb{Z}_2$ (work in progress, G-Izumi).

The extended Haagerup subfactor does not appear to be related to quadratic categories and is the last standing “exotic” subfactor. Its modular data has been computed by Morrison-Gannon.
There remain a number of basic open questions in the subject:

- Do there exist infinite series of Izumi-Haagerup categories whose centers realize the Evans-Gannon modular data? What about analogous series for Asaeda-Haagerup modular data and other families of quadratic categories?

- Does every MTC come from conformal field theory (e.g. as the representation category of a VOA)?

- Two MTC’s are Witt equivalent if their tensor product is the center of some fusion category (Davydov-Mueger-Nikshych-Ostrik). Is the Witt group of (unitary) MTC’s generated by representation categories of quantum groups?

- Is the extended Haagerup subfactor truly exotic? Are there many other exotic MTC’s out there?