Subfactors and Modular Tensor Categories

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Outline

- Motivation
- What is a modular tensor category?
- Where do modular tensor categories come from?
- Some examples of modular tensor categories coming from subfactors.
- Conjectures and questions

Modular Tensor Categories (MTC's) are algebraic structures that arise in a number of related areas of mathematics and mathematical physics.

- Representations of conformal field theories (e.g. rational vertex operator algebras) form MTC's. (Moore-Seiberg, Huang)
- Witten and Reshitikhin-Turaev constructed invariants of links and 3-manifolds and Topogical Quantum Field Theories (TQFT) from MTC's.
- Freedman, Kitaev, Wang have developed a model for quantum computation based on TQFT/MTC - Microsoft Station Q.

What looks like a group but is not a group?

A naive but often fruitful idea in mathematics is to generalize mathematical objects by listing their essential properties as axioms, and then dropping one or more of them.

- By dropping the parallel postulate from Euclid's geometry, one obtains interesting non-Euclidean geometries, such as hyperbolic geometry.
- If one describes the properties of an algebra of functions on a (nice) topological space and drops the requirement that multiplication of functions is commutative, one obtains the notion of a C*-algebra.
- More generally, Connes' theory of noncommutative geometry studies noncommutative analogues of algebras of functions on geometric spaces.

Let G be a finite group, and consider Rep_G, the category of representations of G on finite-dimensional (complex) vector spaces.

Here the objects of Rep_G are representations, and the morphisms are linear maps which intertwine the action of G.

Rep_G is a $\mathbb{C}$-linear Abelian category with finitely many simple objects, and by Maschke's Theorem it is semisimple.

Other properties of Rep_G:

- **Monoidal** - can take $\otimes$; natural isomorphisms

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

satisfying certain coherence relations (*associativity constraint*); trivial representation is tensor identity.

- **Rigid + Spherical** - objects have duals with evaluation and coevaluation maps

$$e_X : X \otimes X^* \to 1, \quad c_X : 1 \to X^* \otimes X$$

satisfying "zig-zag" relations; natural ismorphism $X \to X^{**}$ with left traces equal to right traces.

- **Symmetric** - natural isomorphisms

$$b_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying *the hexagon relations* ("braiding") , such that

$$b_{Y,X} \circ b_{X,Y} = Id_{X \otimes Y}.$$

- (**Unitary** - monoidally equivalent to category of unitary reps, which has C* structure)

Definition

A *fusion category* is a $\mathbb{C}$-linear semisimple rigid monoidal cateory with finitely many simple objects and simple identity object.

We have seen that for a finite group G, Rep_G is a fusion category.

Conversely, every *symmetric* fusion category is equivalent to Rep_G for some finite (super)group G. (Deligne)

What about non-symmetric fusion categories?

Definition

A **braided** fusion category is a fusion category with natural isomorphisms

$$b_{X,Y} : X \otimes Y \to Y \otimes X$$

satifying the *hexagon relations*. (Joyal-Street)

How symmetric is a braiding?

Definition

The **Mueger center** of a braided fusion category $\mathcal{C}$ is

$$\{X \in \mathcal{C} : b_{X,Y} \circ b_{Y,X} = Id_{Y \otimes X}, \ \forall Y\}$$

A braided fusion category is symmetric iff it is equal to its Mueger center.

Definition

A **modular tensor category** is a braided spherical fusion category with trivial Mueger center.

Modular tensor categories have an amazingly intricate structure, first discovered by Moore and Seiberg in the context of conformal field theory.

For a MTC $\mathcal{C}$, one can define S and T matrices as follows. The entries of S are given by the (normalized) values of Hopf links labled by pairs of simple objects of $\mathcal{C}$. T is a diagonal matrix with entries given by twists of the simple objects.

Then S and T are unitary matrices, and the map

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to S, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to T$$

defines a projective representation of the modular group $SL_2(\mathbb{Z})$.

The fusion coefficients $N_{X \otimes Y}^{Z}$ of $\mathcal{C}$ are determined from the S-matrix by the Verlinde formula

$$N_{X \otimes Y}^{Z} = \sum_{r} \frac{S_{X_i, X_r} S_{X_j, X_r} S_{\overline{X_k}, X_r}}{S_{1, X_r}}.$$

Where do modular tensor categories come from?

1. Categories of representations of quantum groups with parameter a root of unity give MTC's (via a semisimplification procedure).
2. Let $\mathcal{C}$ is a monoidal category. The Drinfeld center $Z(\mathcal{C})$ is the monoidal category of *half-braidings* of $\mathcal{C}$ by objects X of $\mathcal{C}$. (Objects of $Z(\mathcal{C})$ are pairs (X, b_X), where b_X is a natural isomorphism $X \otimes Y \to Y \otimes X$ satisfying the braiding relation).

   If $\mathcal{C}$ is a spherical fusion category, then $Z(\mathcal{C})$ is a MTC.

All known examples of MTCs come from these two constructions.

Where do (not-necessarily-modular) fusion categories come from?

1. Representations of (quantum) groups
2. Subfactors

- A **von Neumann algebra** is a *-algebra which is the commutant of a unitary group representation.
- A **factor** is a von Neumann algebra with trivial center.
- A **subfactor** is a unital inclusion $N \subseteq M$ of (∞-dim, finite-trace) factors.

Let $N \subseteq M$ be a subfactor.

The **index** $[M : N]$ of a subfactor is the Murray-von Neumann coupling constant of ${}_N L^2(M)$.

Theorem (Jones '83)

$[M : N] \in \{4cos^2 \frac{\pi}{k}\}_{k=3,4,5...} \cup [4, \infty]$.

Given a finite index subfactor $N \subseteq M$, one can consider the category $\mathcal{N}$ of $N - N$ bimodules $\otimes$-generated by ${}_N M_N$.

- $\mathcal{N}$ is a semisimple rigid monoidal category, but **not** necessarily finite.
- If $\mathcal{N}$ is finite, it is a fusion category.
- **Every** unitary fusion category can be realized this wayas a subcategory of such an $\mathcal{N}$.

Subfactors with index $< 4 \implies$ fusion categories associated to quantum $SU(2)$.

In the 1990's, 2 "exotic" subfactors were discovered with index slightly above 4

1. Haagerup subfactor (index $\frac{5+\sqrt{13}}{2}$)
2. Asaeda-Haagerup subfactor (index $\frac{5+\sqrt{17}}{2}$)

More recently, the extended Haagerup subfactor was constructed by Bigelow-Morrison-Peters-Snyder (non-quadratic index).

Let $\mathcal{C}$ be a fusion category. Let $Inv(\mathcal{C})$ be the tensor subcategory of invertible objects.

On a decategorified level, $Inv(\mathcal{C})$ gives a finite group G, which acts on the set of ($\cong$-classes of) simple objects of $\mathcal{C}$.

Definition

A fusion category $\mathcal{C}$ is **pointed** if $\mathcal{C} = Inv(\mathcal{C})$.

Every pointed fusion category is equivalent to Vec_G^ω, the category of G-graded vector spaces, with associator given by $\omega \in H^3(G, \mathbb{C}^*)$.

Definition

A fusion category $\mathcal{C}$ is **quadratic** if there are exactly two orbits of simple objects under the action of $Inv(\mathcal{C})$.

Ex: The fusion category associated to the Haagerup subfactor has simple objects

$$\{g\}_{g \in G} \cup \{gX\}_{g \in G}$$

with fusion rules

$$g \cdot h = gh, \quad gX = Xg^{-1}, \quad X^2 = 1 + \sum_{g \in G} gX.$$

where $G = \mathbb{Z}_3$.

A category with fusion rules as above for some finite Abelian group G is called an Izumi-Haagerup category.

In the 1990's, Izumi introduced a general method for constructing quadratic fusion categories from endomorphisms of C^* algebras.

- Very explicit description of the categories - allows one to perform computations in the Drinfeld center.
- Explicit polynomial equations whose solutions give Izumi-Haagerup categories for groups of odd order.
- Solved the equations for $\mathbb{Z}_3$ and $\mathbb{Z}_5$ and found the modular data of the Drinfeld centers.

More recently, Evans and Gannon solved Izumi's equations for many more cyclic groups of odd order and gave a general formula for modular data associated to Izumi-Haagerup subfactors for cyclic group of odd order.

This modular data appears to be a "graft" of modular data associated to dihedral groups and to certain quantum groups.

Based on this Evans-Gannon argued that the Haagerup subfactor should not be viewed as "exotic" at all.

However there is still no general construction for Izumi-Haagerup categories, and it is not known whether Evans-Gannon's infinite series of modular data is realized by a corresponding series of MTC's.

It turns out the Asaeda-Haagerup subfactor is also related to an Izumi-Haagerup category, but in a more complicated way: its fusion category is Morita equivalent to a $\mathbb{Z}_2$ orbifold of an I-H category for the group $\mathbb{Z}_4 \times \mathbb{Z}_2$ (G-Izumi-Snyder).

The modular data of the Asaeda-Haagerup subfactor has a similar structure to the Evans-Gannon modular data, and can be generalized to a series for the groups $\mathbb{Z}_{4n} \times \mathbb{Z}_2$ (work in progress, G-Izumi).

The extended Haagerup subfactor does not appear to be related to quadratic categories and is the last standing "exotic" subfactor. Its modular data has been computed by Morrison-Gannon.

There remain a number of basic open questions in the subject:

- Do there exist infinite series of Izumi-Haagerup categories whose centers realize the Evans-Gannon modular data? What about analogous series for Asaeda-Haagerup modular data and other families of quadratic categories?
- Does every MTC come from conformal field theory (e.g. as the representation category of a VOA)?
- Two MTC's are Witt equivalent if their tensor product is the center of some fusion category (Davydov-Mueger-Nikshych-Ostrik). Is the Witt group of (unitary) MTC's generated by representation categories of quantum groups?
- Is the extended Haagerup subfactor truly exotic? Are there many other exotic MTC's out there?