Bulk-edge correspondence in the presence of a mobility gap

Gian Michele Graf
ETH Zurich

Topological Matter, Strings, K-theory and related areas
IGA/AMSI Workshop
26-30 September 2016
Adelaide

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based on joint work with A. Elgart, J. Schenker; J. Shapiro

Outline

Goals of the talk

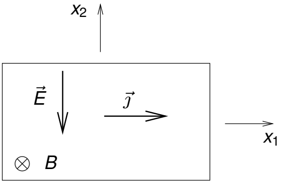
- Difference between spectral and mobility gap
- Bulk-edge correspondence for quantum Hall Hamiltonians (2 dim)
- Bulk-edge correspondence for chiral Hamiltonians (1 dim)

Goal of the talk

Quantum Hall systems

Chiral systems

The experiment (von Klitzing, 1980)



B: magnetic field
$\vec{E}$: electric field
$\vec{\jmath}$: current density

Electron gas of
- density n
- Fermi energy μ

Hall-Ohm law

$$\vec{\jmath} = \underline{\sigma}\vec{E}\,, \qquad \underline{\sigma} = \begin{pmatrix} \sigma_{\mathrm{D}} & \sigma_{\mathrm{H}} \\ -\sigma_{\mathrm{H}} & \sigma_{\mathrm{D}} \end{pmatrix}$$

σ_{H}: Hall conductance
σ_{D}: ohmic (dissipative) conductance

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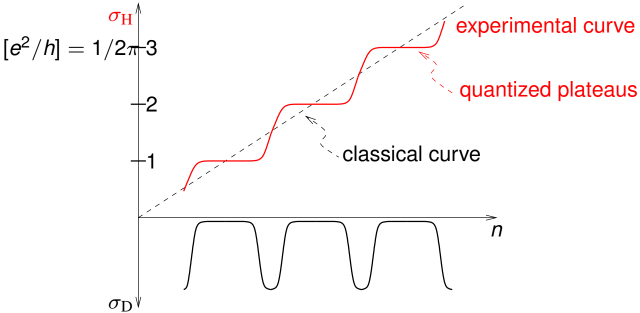

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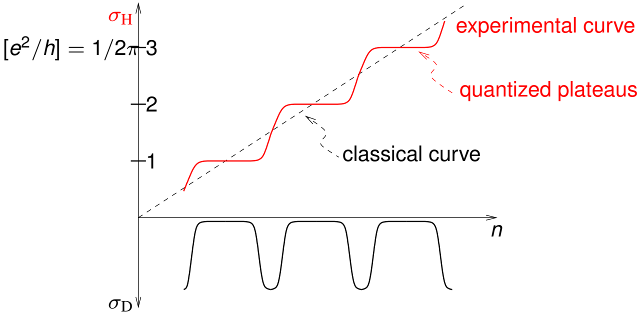
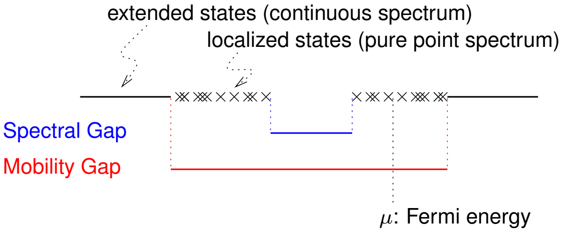


Width of plateaus increases with disorder

Spectral vs. Mobility Gap

The spectrum of a single-particle Hamiltonian



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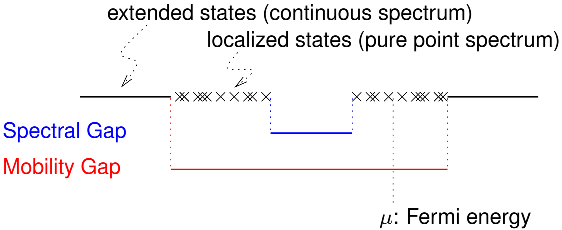

- (integrated) density of states $n(\mu)$ is constant for μ in a Spectral Gap, and strictly increasing otherwise

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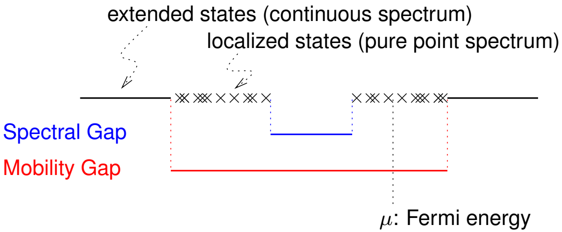

- (integrated) density of states $n(\mu)$ is constant for μ in a Spectral Gap, and strictly increasing otherwise
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Spectral vs. Mobility Gap

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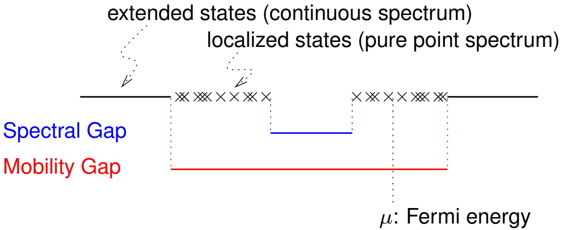

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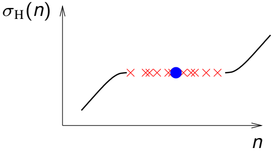

Plateaus arise because of a Mobility Gap only!

Mobility gap, technically speaking

Hamiltonian H_B on $\ell^2(\mathbb{Z}^d)$
$P_\mu = E_{(-\infty,\mu)}(H_B)$ Fermi projection,

Assumption. Fermi projection has strong off-diagonal decay:

$$\sup_{x'} \mathrm{e}^{-\varepsilon|x'|} \sum_x \mathrm{e}^{\nu|x-x'|} |P_\mu(x,x')| < \infty$$

(some $\nu > 0$, all $\varepsilon > 0$)

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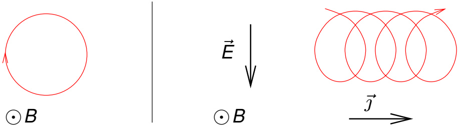
- Trivially true for H_B a multiplication operator in position space
- Trivially false for H_B a function of momentum ($P_\mu(x,0) \sim |x|^{-d}$)
- Proven in (virtually) all cases where localization is known.

IQHE as a Bulk effect

Paradigm: Cyclotron orbit drifting under a electric field $\vec{E}$

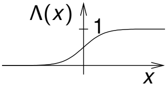


Hamiltonian H_B in the plane. Kubo formula (linear response to $\vec{E}$)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2] \big]$$

where

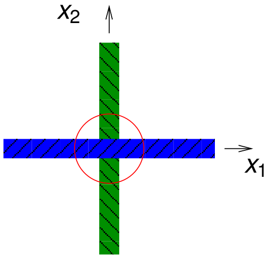
$\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches



IQHE as a Bulk effect (remarks)

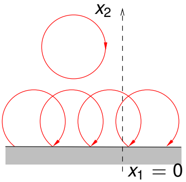
$$\sigma_{\rm B} = {\rm i}\, \mathrm{tr} P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2] \big]$$

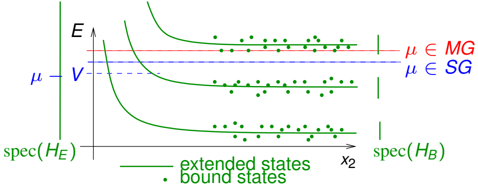
where $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches. Supports of $\vec{\nabla}\Lambda_i$:



Remark. The trace is well-defined. Roughly: An operator has a well-defined trace if it acts non-trivially on finitely many states only. Here the intersection contains only finitely many sites.

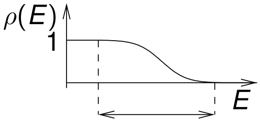
IQHE as an edge effect (spectral gap)





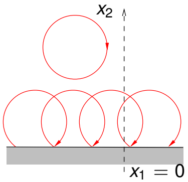
Hamiltonian H_E on the upper half-plane: restriction of H_B through boundary conditions at $x_2 = 0$.

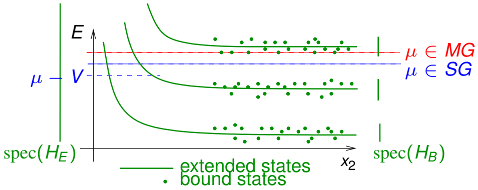
State $\rho(H_E)$: 1-particle density matrix, e.g. $\rho(H_E) = E_{(-\infty,\mu)}(H_E)$, or (actually) smooth



supp $\rho' \subset$ Spectral Gap for H_B (not for H_E)

IQHE as an edge effect (spectral gap)





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Current operator across $x_1 = 0$: $\mathrm{i}[H_E, \Lambda_1]$

$$I = \mathrm{i}\ \mathrm{tr}(\rho(H_E + V) - \rho(H_E))[H_E, \Lambda_1]$$

As $V \to 0$: $I/V \to \sigma_\mathrm{E}$

$$\sigma_\mathrm{E} = \mathrm{i}\ \mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

Equality of conductances

Theorem (Schulz-Baldes, Kellendonk, Richter). Ergodic setting. If the Fermi energy μ lies in a Spectral Gap of H_B, then

$$\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}.$$

In particular, σ_{E} does not depend on ρ', nor on boundary conditions.

What about the case of a Mobility Gap?

Is

$$\sigma_{\mathrm{E}} = -\mathrm{i}\, \mathrm{tr}(\rho'(H_E)[H_E, \Lambda_1])$$

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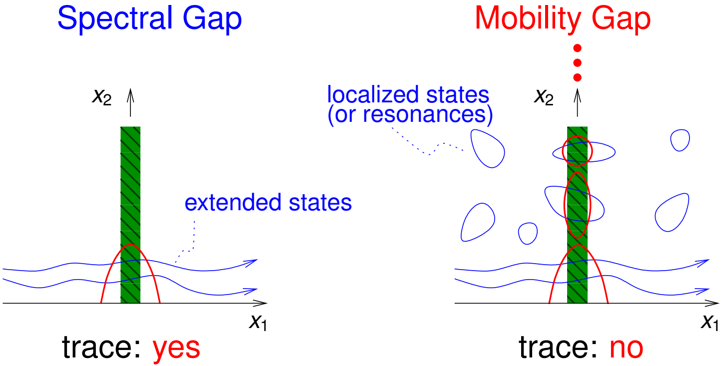
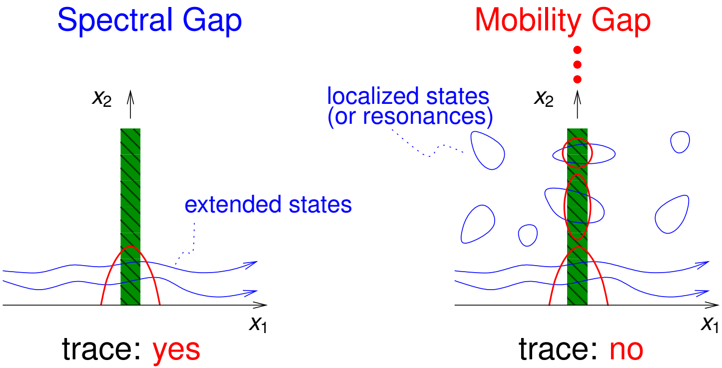


∴ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

What about the case of a Mobility Gap?



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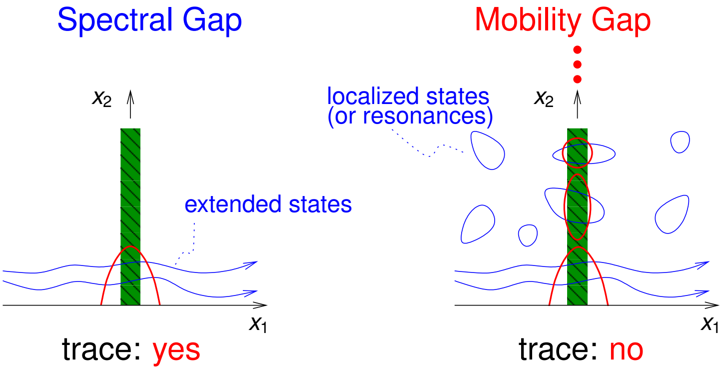
Guiding principle: Localized states should not contribute to the edge current

What about the case of a Mobility Gap?



$\therefore$ the definition of σ_{E} needs to be changed in case of a Mobility Gap!

Analogy: Electrodynamics of continuous media

$$\vec{\jmath} = \vec{\jmath}_F + \mathrm{curl}\,\vec{M} \equiv \text{free} + \text{molecular currents}$$

Localized states should not contribute to the (free) edge current

Equality of conductances

For a suitable definition of $\sigma_{\rm E}$:

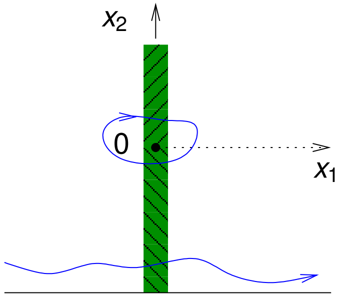
Theorem (Elgart, G., Schenker). If supp ρ' lies in a Mobility Gap, then

$$\sigma_{\rm E} = \sigma_{\rm B}$$

In particular $\sigma_{\rm E}$ does not depend on ρ', nor on boundary conditions.

Definition of σ_{E} in case of a Mobility Gap

Replace H_E to H_a ($a > 0$) as follows



edge: $x_2 = -a$
(eventually: $-a \to -\infty$)

- Current across the portion of $x_1 = 0$:

$$-\mathrm{i}\,\mathrm{tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2) \quad \text{(exists!)}$$

- Current across the portion :

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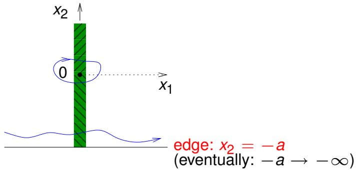

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$$\rho'(H_a) \rightsquigarrow \rho'(H_B) = \sum_{\lambda} \rho'(\lambda)\psi_\lambda(\psi_\lambda, \cdot)$$

(sum over eigenvalues λ of H_B: $H_B\psi_\lambda = \lambda\psi_\lambda$)

$$(\psi_\lambda, [H_B, \Lambda_1](1 - \Lambda_2)\psi_\lambda) = -(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda)$$

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- Together:

$$\sigma_{\mathrm{E}} = \lim_{a\to\infty} -\mathrm{i}\,\mathrm{tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2) + \\ + \mathrm{i}\sum_\lambda \rho'(\lambda)(\psi_\lambda, [H_B, \Lambda_1]\Lambda_2\psi_\lambda)$$

Sketch of proof of $\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}$

Technical tool: Representation of $\rho(H_a)$ by

- quasi-analytic extension $\rho(z)$, $(z = x + \mathrm{i}y \in \mathbb{C})$
- resolvent $R(z) = (H_a - z)^{-1}$

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$$\rho(H_a) = \frac{1}{2\pi} \int_{\mathbb{C}} d^2 z \, \partial_{\bar{z}} \rho(z) R(z)$$

with $d^2 z = dxdy$, $\partial_{\bar{z}} = \partial_x + \mathrm{i}\partial_y$.

Note: $\partial_{\bar{z}} \rho(z)$ supported near $\operatorname{supp} \rho \subset (-\infty, 0] \subset \mathbb{C}$

Sketch of proof

$$R(z) = (H_a - z)^{-1}$$

$$\rho(H_a) = \frac{1}{2\pi}\int d^2z\,\partial_{\bar{z}}\rho(z)R(z)$$

$$\rho'(H_a) = -\frac{1}{2\pi}\int d^2z\,\partial_{\bar{z}}\rho(z)R(z)^2$$

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$$\rho'(H_a)[H_a, \Lambda_1] = -\frac{1}{2\pi} \int d^2z \, \partial_{\bar{z}} \rho(z) R(z)^2 [H_a, \Lambda_1]$$
$$[\rho(H_a), \Lambda_1] = -\frac{1}{2\pi} \int d^2z \, \partial_{\bar{z}} \rho(z) R(z) [H_a, \Lambda_1] R(z)$$

Sketch of proof

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$$\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 = -\frac{1}{2\pi}\int d^2z\,\partial_{\bar z}\rho(z)R(z)^2[H_a,\Lambda_1]\Lambda_2$$
$$[\rho(H_a),\Lambda_1]\Lambda_2 = -\frac{1}{2\pi}\int d^2z\,\partial_{\bar z}\rho(z)R(z)[H_a,\Lambda_1]R(z)\Lambda_2$$

Sketch of proof

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Sketch of proof

$$\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 \neq -\frac{1}{2\pi}\int d^2z\, \partial_{\bar{z}}\rho(z) R(z)[H_a, \Lambda_1]\Lambda_2 R(z)$$

$$[\rho(H_a), \Lambda_1]\Lambda_2 = -\frac{1}{2\pi}\int d^2z\, \partial_{\bar{z}}\rho(z) R(z)[H_a, \Lambda_1]R(z)\Lambda_2$$

- In first equation (RHS), move one power of $R(z)$ to the far right. Difference is $[R(z), R(z)[H_a, \Lambda_1]\Lambda_2]$

Sketch of proof

$$\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 \neq -\frac{1}{2\pi}\int d^2z\,\partial_{\bar z}\rho(z)R(z)[H_a,\Lambda_1]\Lambda_2 R(z)$$
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- Second equation (LHS) is $[\rho(H_a)\Lambda_2,\Lambda_1]$

Sketch of proof

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- Second equation (LHS) is $[\rho(H_a)\Lambda_2,\Lambda_1]$
- Difference involves $\Lambda_2 R(z) - R(z)\Lambda_2 = [\Lambda_2, R(z)] = R(z)[H_a,\Lambda_2]R(z)$

The poor man's non-commutative geometry

$$\operatorname{tr}[A, B] = 0 \qquad \leftrightsquigarrow \qquad \int f'(x)dx = 0$$

$$(AB,\ BA \text{ trace class}) \qquad\qquad (\operatorname{supp} f \text{ compact})$$

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For $f = \chi_{(-\infty,0]} \cdot g$ we have $f' = -\delta \cdot g + \chi_{(-\infty,0]} \cdot g'$ and

$$g(0) = \int_{-\infty}^{0} g'(x)dx$$

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$\therefore$ To add the trace of a commutator is to apply a non-commutative Stokes Theorem $\int_{\partial X} g = \int_X dg$

Picture of proof of $\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}$

To add a commutator is $\int_{\partial X} g = \int_X dg$

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Let X be the non-commutative space (x_1, x_2, E).

Picture of proof of $\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}$

To add a commutator is $\int_{\partial X} g = \int_X dg$

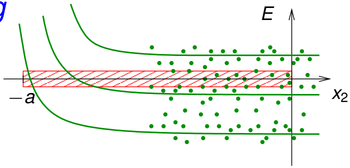
Let X be the non-commutative space (x_1, x_2, E). Shown plane $x_1 = 0$

Picture of proof of $\sigma_{\mathrm{E}} = \sigma_{\mathrm{B}}$

To add a commutator is $\int_{\partial X} g = \int_X dg$

- Definition of σ_{E} is

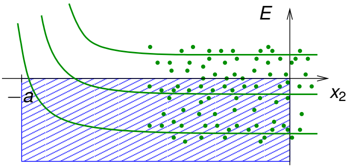
$$\sigma_{\mathrm{E}} + \text{spurious} := -\mathrm{i} \lim_{a\to\infty} \operatorname{tr} \rho'(H_a)[H_a, \Lambda_1]\Lambda_2$$



- Add

$$0 = \operatorname{tr}([R(z), R(z)[H_a, \Lambda_1]\Lambda_2])$$

$(z \in \mathbb{C} \text{ near } (-\infty, 0])$

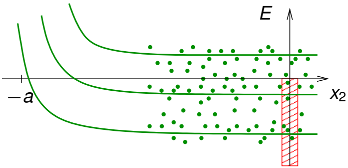


- Add

$$0 = \operatorname{tr}([\rho(H_a)\Lambda_2, \Lambda_1])$$

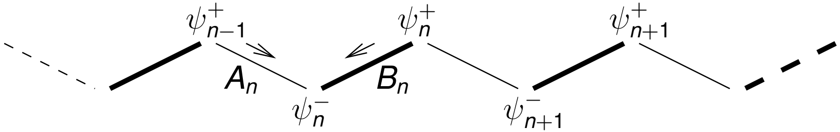
The operator is supported in the bulk, and equals

$$\sigma_{\mathrm{B}} + \text{spurious}$$



The model (1 dimensional)

Alternating chain with nearest neighbor hopping



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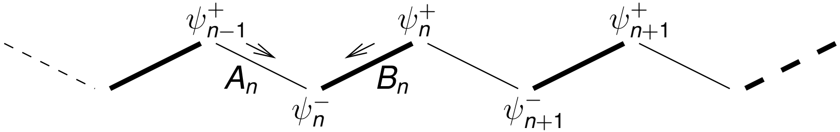

Hilbert space: sites arranged in dimers

$$\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^N) \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix}_{n \in \mathbb{Z}}$$

Hamiltonian

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

with S, S^* acting on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$ as

$$(S\psi^+)_n = A_n \psi^+_{n-1} + B_n \psi^+_n , \qquad (S^*\psi^-)_n = A^*_{n+1} \psi^-_{n+1} + B^*_n \psi^-_n$$

($A_n, B_n \in \mathrm{GL}(N)$ almost surely)

Chiral symmetry

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{H, \Pi\} \equiv H\Pi + \Pi H = 0$$

hence

$$E_I(H)\Pi + \Pi E_{-I}(H) = 0 \qquad (E_I(H) \text{ spectral projection for } I \subset \mathbb{R})$$

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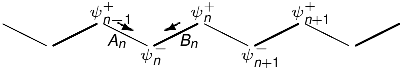


- Eigenvalue equation $H\psi = \lambda\psi$ is $S\psi^+ = \lambda\psi^-$, $S^*\psi^- = \lambda\psi^+$, i.e.

$$A_n\psi^+_{n-1} + B_n\psi^+_n = \lambda\psi^-_n\,, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = \lambda\psi^+_n$$

is one 2nd order difference equation, but two 1st order for $\lambda = 0$

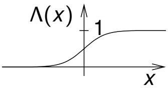
Bulk index

Let

$$\Sigma = \operatorname{sgn} H$$

Definition. The Bulk index is

$$\mathcal{N} = \frac{1}{2}\operatorname{tr}(\Pi\Sigma[\Lambda, \Sigma])$$



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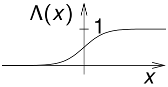

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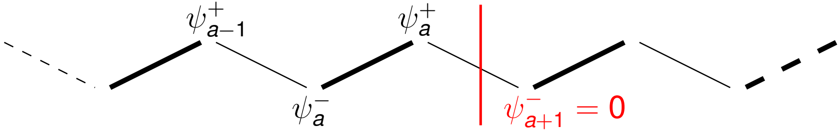
Equivalently

$$-\mathcal{N} = \operatorname{tr}(\Pi P_+[\Lambda, P_-]) + \operatorname{tr}(\Pi P_-[\Lambda, P_+])$$

using $P_+ := E_{(0,+\infty)}$, $P_- := E_{(-\infty,0)}$ and $\Sigma = P_+ - P_-$

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \leq a$ (Dirichlet boundary condition $\psi^-_{a+1} = 0$). Chiral symmetry preserved.

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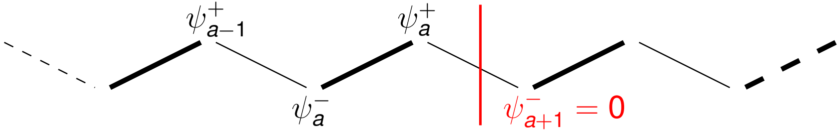

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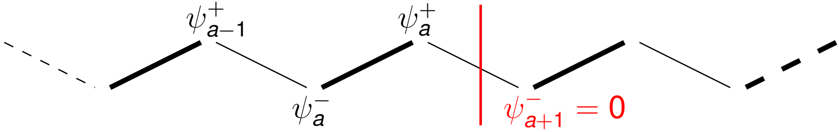

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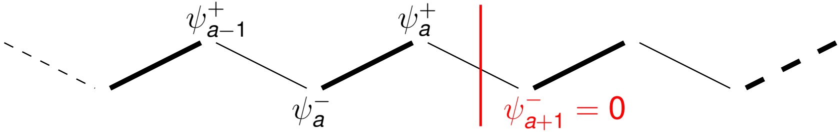

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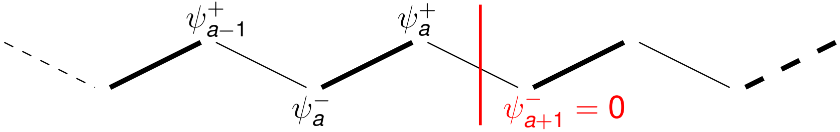
Definition. The Edge index is

$$\mathcal{N}_a = \mathcal{N}^+_a - \mathcal{N}^-_a = \mathrm{tr}(\Pi P_{0,a})$$

A vanishing lemma



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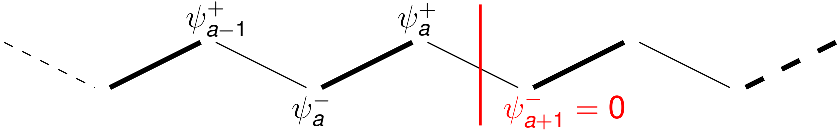

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$$A_n\psi^+_{n-1} + B_n\psi^+_n = 0, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = 0$$

Lemma.

$$\mathcal{N}_a^+ = \dim\{\psi^+ : \mathbb{Z} \to \mathbb{C}^N \mid S\psi^+ = 0, \psi^+_n \text{ is } \ell^2 \text{ at } n \to -\infty\}$$
$$\mathcal{N}_a^- = 0$$

In particular $\mathcal{N}_a$ is independent of a.

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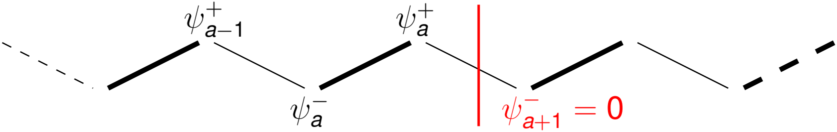

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In particular $\mathcal{N}_a$ is independent of a. Call it $\mathcal{N}^\sharp$.

Bulk-edge duality

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

$$\mathcal{N} = \mathcal{N}^\sharp$$

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Remark. Consider the dynamical system $A_n \psi^+_{n-1} + B_n \psi^+_n = 0$ with Lyaponov exponents

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Phase boundaries correspond to $\gamma_i = 0$ (cf. Prodan et al.)

Proof

Recall $\mathcal{N}_a = \mathrm{tr}(\Pi P_{0,a})$

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Proof of Theorem. On the Hilbert space $\mathcal{H}_a$ corresponding to $n \leq a$

$$\operatorname{tr}(\Pi \Lambda) = N\Big(\sum_{n \leq a} \Lambda(n)\Big) \operatorname{tr}_{\mathbb{C}^2} \Pi = 0$$

though $\|\Pi\Lambda\|_1 = \|\Lambda\|_1 \to \infty$, $(a \to +\infty)$

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$$\operatorname{tr}(\Pi \Lambda) = 0$$

So,

$$\operatorname{tr}(\Pi\Lambda) = \underbrace{\operatorname{tr}(\Pi\Lambda P_{0,a})}_{\to \mathcal{N}^\sharp} + \underbrace{\operatorname{tr}(\Pi\Lambda P_{+,a}) + \operatorname{tr}(\Pi\Lambda P_{-,a})}_{\to \operatorname{tr}(\Pi P_-[\Lambda, P_+]) + \operatorname{tr}(\Pi P_+[\Lambda, P_-]) = -\mathcal{N}}$$

q.e.d.

Summary

Elementary methods used to establish bulk-edge correspondence in simple models of topological insulators in presence of a mobility gap