Gerbes and time-reversal-invariant topological insulators

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Partly based on the joint work with David Carpentier, Pierre Delplace, Michel Fruchart and Clément Tauber [15, 16]. For a more complete account see [18].

I. PLAN

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II. BUNDLE GERBES WITH CONNECTION AND THEIR HOLONOMY

- Bundle gerbes - an example of higher structures: 1-degree higher as line bundles
- Introduced by M. K. Murray [1] in 1996 as geometric examples of more abstract gerbes of J. Giraud [2] and J.-L. Brylinski [3].
- They were applied in physics e.g. to describe topological Wess-Zumino amplitudes in conformal field theory in line with earlier works of O. Alvarez [4] and K.G. [5] that used a cohomological language.
- Topic of the talk: a relation of gerbes to the topological insulators.

Definition. A bundle *gerbe* $\mathcal{G}$ with unitary connection (below, *gerbe* for short) over manifold M is a triple $(Y, B, \mathcal{L})$ s.t.
- Y is a manifold equipped with a surjective submersion $\pi : Y \to M$
- B is a 2-form on Y
- $\mathcal{L}$ is a line bundle (always with hermitian structure and unitary connection) over $Y^{[2]} \equiv Y \times_M Y \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y$ with the curvature $p_2^*B - p_1^*B$
- $\mathcal{L}$ comes with a groupoid multiplication $\mathcal{L}_{(y_1,y_2)} \times \mathcal{L}_{(y_2,y_3)} \longrightarrow \mathcal{L}_{(y_1,y_3)}$ respecting the structure of $\mathcal{L}$.

- Necessarily, $dB = \pi^*H$ where H is a closed 3-form on M called the *curvature* of the gerbe $\mathcal{G}$.
- Gerbes over M form a 2-category with 1-morphisms $\eta : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ and 2-morphisms $\mu : \eta_1 \longrightarrow \eta_2$ between a pair of 1-morphisms $\eta : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ [6].
- As line bundles, gerbes may be tensored dualized or pulled back.
- The abelian group (under $\otimes$) $\mathbb{G}(M)$ of 1-isomorphism classes of gerbes over M is isomorphic to the (real) Deligne hyper-cohomology group $\mathbb{H}^3(M)$ [8] and to the group $\widehat{H}^3(M)$ of Cheeger-Simons differential characters [9].
- If Σ is an oriented closed 2-surface and $\phi : \Sigma \longrightarrow M$ is a smooth then for any gerbe $\mathcal{G}$ over M,
$$[\phi^*\mathcal{G}] \in \mathbb{G}(\Sigma) = U(1).$$
The corresponding phase in $U(1)$ is called the holonomy of $\mathcal{G}$ along the map ϕ and is denoted $Hol_{\mathcal{G}}(\phi)$. Physicists' name for $Hol_{\mathcal{G}}(\phi)$ is the Wess-Zumino amplitude $\mathrm{e}^{\mathrm{i}S_{WZ}(\phi)}$ of ϕ [10].
- If there exists an extension of ϕ to a map $\widetilde{\phi} : \widetilde{\Sigma} \longrightarrow M$ from an oriented 3-manifold $\widetilde{\Sigma}$ with the boundary $\partial\widetilde{\Sigma} = \Sigma$ then
$$Hol_{\mathcal{G}}(\phi) \,=\, \exp\Big[\mathrm{i}\int_{\widetilde{\Sigma}} \widetilde{\phi}^*H\Big].$$

III. EXAMPLE: BASIC GERBE OVER $U(N)$

- Let $M = U(N)$ and $H = \frac{1}{12\pi}\,\mathrm{tr}(u^{-1}du)^3$ be the closed bi-invariant 3-form on $U(N)$.
- A gerbe $\mathcal{G}$ on $U(N)$ with curvature H is called *basic*. It is unique up to 1-isomorphisms.
- A convenient construction of such a gerbe exploits the ambiguities in taking the logarithm of a unitary matrix. It is essentially due to Murray-Stevenson [7].
- In this construction, $\mathcal{G} = (Y, B, \mathcal{L})$ where
  - $Y = \big\{(\epsilon, u) \in]-2\pi, 0[\times U(N) \mid \mathrm{e}^{-\mathrm{i}\epsilon} \notin spec(u)\big\}$ with $\pi : Y \longrightarrow U(N)$ forgetting ϵ
  - B such that $dB = \pi^* H$ is defined from the Poincaré Lemma using the homotopy $\chi : [0, 2\pi]\times Y \longrightarrow Y$

$$\chi(t, \epsilon, u) = (\epsilon, \mathrm{e}^{-\mathrm{i}th_\epsilon(u)})$$

    where $h_\epsilon(u) = \frac{\mathrm{i}}{2\pi}\ln_{-\epsilon}(u)$ with the the values of $\ln_{-\epsilon}(z)$ in $\mathbb{R}\times \mathrm{i}]-\epsilon-2\pi, -\epsilon[$
  - For $\epsilon \leq \epsilon'$,

$$h_\epsilon(u) - h_{\epsilon'}(u) = p_{\epsilon,\epsilon'}(u)$$

    where $p_{\epsilon,\epsilon'}(u)$ is the spectral projector of u on the subspace $E_{\epsilon,\epsilon'}(u) \subset \mathbb{C}^N$ corresponding to the eigenvalues $\mathrm{e}^{-\mathrm{i}e_n}$ with $\epsilon < e_n < \epsilon'$. One takes

$$\mathcal{L}_{(\epsilon,\epsilon',u)} \;=\; \wedge^{max} E_{\epsilon,\epsilon'}(u)$$

    for the fiber of line bundle $\mathcal{L}$ over $(\epsilon, \epsilon', u) \in Y^{[2]}$
  - The connection on $\mathcal{L}$ is essentially the Berry one (modified by the addition of a 1-form)
  - The groupoid multiplication on $\mathcal{L}$ is induced by the isomorphism

$$\wedge^{max} E_{\epsilon,\epsilon'}(u) \;\otimes\; \wedge^{max} E_{\epsilon',\epsilon''}(u) \;\cong\; \wedge^{max} E_{\epsilon,\epsilon''}(u)$$

    for $\epsilon \leq \epsilon' \leq \epsilon''$.

IV. SQUARE ROOT OF THE GERBE HOLONOMY

Suppose that $\mathcal{G}$ is a gerbe over M with curvature H and $\Theta : M \longrightarrow M$ is an involution preserving H.

Definition (...,[11],...) A Θ-equivariant structure on $\mathcal{G}$ is composed of
- a 1-isomorphism $\eta : \mathcal{G} \longrightarrow \Theta^*\mathcal{G}$
- a 2-isomorphism $\mu : \Theta^*\eta \circ \eta \longrightarrow Id_{\mathcal{G}}$ between 1-isomorphisms of gerbe $\mathcal{G}$ s.t.
- μ is Θ-invariant (i.e. $Id_\eta \circ \mu = \Theta^*\mu \circ Id_\eta$ as 2-isomorphisms between the 1-isomorphisms $\eta \circ \Theta^*\eta \circ \eta : \mathcal{G} \longrightarrow \Theta^*\mathcal{G}$ and $\eta : \mathcal{G} \longrightarrow \Theta^*\mathcal{G}$).

- We shall call a gerbe $\mathcal{G}$ over M equipped with a Θ-equivariant structure a Θ-gerbe.
- Let $\vartheta : \Sigma \longrightarrow \Sigma$ be an *orientation-preserving* map with a discrete set of fixed points.
  Example: for the 2-torus $\mathbb{R}^2/(2\pi\mathbb{Z}^2) \equiv \mathbb{T}^2$ we take ϑ generated by $k \mapsto -k$ for $k \in \mathbb{R}^2$.

Proposition. Let $\phi : (\Sigma, \theta) \longrightarrow (M, \Theta)$ (i.e. ϕ is equivariant: $\phi \circ \vartheta = \Theta \circ \phi$). Assume that the fixed point set $M^\Theta \subset M$ of Θ is 1-connected. Then a Θ-equivariant structure on a gerbe $\mathcal{G}$ over M permits to define to a unique square root $\sqrt{Hol_{\mathcal{G}}(\phi)}$ of the holonomy of $\mathcal{G}$ along ϕ.

- If there exists an extension $\widetilde{\phi} : (\widetilde{\Sigma}, \widetilde{\theta}) \longrightarrow (M, \Theta)$ of ϕ for an *orientation-preserving* involution $\widetilde{\vartheta} : \widetilde{\Sigma} \longrightarrow \widetilde{\Sigma}$ reducing to ϑ on $\partial\widetilde{\Sigma} = \Sigma$ then

$$\sqrt{Hol_{\mathcal{G}}(\phi)} \;=\; \exp\Big[\tfrac{\mathrm{i}}{2}\int_{\widetilde{\Sigma}} \widetilde{\phi}^* H\Big].$$

V. A $3d$ INDEX

- Let R be an oriented compact 3-manifold without boundary and $\rho : R \longrightarrow R$ an *orientation-reversing* involution with a discrete set of fixed points.
  Example: for the 3-torus $\mathbb{R}^3/(2\pi\mathbb{Z}^3) \equiv \mathbb{T}^3$ we take ρ generated by $k \mapsto -k$ for $k \in \mathbb{R}^3$.
- Let $F \subset R$ be the closure of a fundamental domain for ρ that is a submanifold with boundary of R. Then ρ preserves ∂F together with its orientation inherited from R.
  Example: for $R = \mathbb{T}^3$ with ρ as above we may take $F = [0, \pi] \times \mathbb{T}^2$ with ∂F composed of two connected components: $\{\pi\} \times \mathbb{T}^2 \equiv \mathbb{T}^2_\pi$ and $\{0\} \times \mathbb{T}^2 \equiv \mathbb{T}^2_0$.

Proposition. Let $\mathcal{G}$ be a Θ-gerbe over M with curvature H and $\Phi : (R, \rho) \longrightarrow (M, \Theta)$.
If $M^\Theta \subset M$ is 1-connected then the ratio

$$\frac{\exp\Big[\frac{\mathrm{i}}{2}\int\limits_F \Phi^* H\Big]}{\sqrt{Hol_{\mathcal{G}}(\Phi|_{\partial F})}} \;\equiv\; \mathcal{K}_{\mathcal{G}}(\Phi)$$

taking the values ± 1 is independent of the choice of the fundamental domain $F \subset R$.

Remark. The proof of Proposition relies on local expressions for $\sqrt{Hol_{\mathcal{G}}(\phi)}$ provided by gerbes or the cohomological approach of [5].

VI. TIME-REVERSAL ON $U(N)$

- In quantum mechanics with the space of states $\mathbb{C}^N$, the time reversal is realized by an anti-unitary map $\theta : \mathbb{C}^N \longrightarrow \mathbb{C}$ such that $\theta^2 = \pm I$ (with N necessarily even for the minus sign)..
- In both cases, θ induces an involution $\Theta : U(N) \longrightarrow U(N)$ by the formula $\Theta(u) = \theta u \theta^{-1}$ and $\Theta^* H = H$ for the bi-invariant 3-form H considered above.

Proposition. 1. If $\theta^2 = I$ then $\exists$ a Θ-equivariant structure on the basic gerbe $\mathcal{G}$ over $U(N)$. However, in this case $U(N)^\Theta \cong O(N)$ is not 1-connected.
2. If $\theta^2 = -I$ then $\exists$ **no** Θ-equivariant structure on the basic gerbe $\mathcal{G}$ over $U(N)$. However Θ lifts to the involution $\widehat{\Theta}$ on the double cover $\widehat{U}(N)$ of $U(N)$ and $\exists$ a $\widehat{\Theta}$-equivariant structure on the pullback $\widehat{\mathcal{G}}$ to $\widehat{U}(N)$ of the basic gerbe over $U(N)$. The fixed point set $\widehat{U}(N)^{\widehat{\Theta}} \cong Sp(N) \sqcup Sp(N)$ is simply connected but not connected.

- For $\theta^2 = I$ the lack of 1-connectivity of $U(N)^\Theta$ does not allow to define the square root $\sqrt{Hol_{\mathcal{G}}(\phi)}$ nor of the $3d$ index $\mathcal{K}(\Phi)$ for equivariant maps ϕ and Φ.
- For $\theta^2 = -I$, every map $\phi : (\mathbb{T}^2, \vartheta) \longrightarrow (U(N), \Theta)$ and every map $\Phi : (\mathbb{T}^3, \rho) \longrightarrow (U(N), \Theta)$ may be lifted to $\widehat{\phi} : (\mathbb{T}^2, \theta) \longrightarrow (\widehat{U}(N), \widetilde{\Theta})$ and $\widehat{\Phi} : (\mathbb{T}^3, \theta) \longrightarrow (\widehat{U}(N), \widetilde{\Theta})$, respectively, and one can still define uniquely $\sqrt{Hol_{\widehat{\mathcal{G}}}(\widehat{\phi})}$ and $\mathcal{K}(\widehat{\Phi})$ in spite of the lack of connectivity of $\widehat{U}(N)^{\widetilde{\Theta}}$. Besides, these quantities do not depend on the choice of the lifts $\widehat{\phi}$ and $\widehat{\Phi}$. We shall use the notation $\sqrt{Hol_{\mathcal{G}}(\phi)}$ and $\mathcal{K}(\Phi)$ for them.

Remark. 1. The last point does not hold for all $(\Sigma.\vartheta)$ and (R, ρ).
2. The obstruction to the existence of Θ-equivariant structure for $\theta^2 = -I$ is the non-triviality of the flat line bundle over $U(N)$

$$Q = Y \times \mathbb{C}/\sim \qquad \text{where} \qquad (\epsilon, u, z) \sim (\epsilon', u, (-1)^{\dim(\mathrm{E}_{\epsilon,\epsilon'}(\mathrm{u}))} z) \tag{1}$$

that excludes the existence of 2-isomorphism μ.

VII. APPLICATION TO TOPOLOGICAL INSULATORS

- In the simplest case, the d-dimensional insulators are described by lattice Hamiltonians that, after the discrete Fourier-Bloch transformation, give rise to a map
$$\mathbb{T}^d \ni k \;\longmapsto\; h(k) = h(k)^\dagger \;\in\; End(\mathbb{C}^N)$$
and all the hermitian matrices $h(k)$ have a spectral gap around the Fermi energy ϵ_F. Denote by $p(k)$ the spectral projectors on the eigenstates of $h(k)$ with energies $< \epsilon_F$.
- For the fermionic time-reversal symmetric insulators,
$$\theta h(k)\theta^{-1} = h(-k) \qquad \text{and} \qquad \theta p(k)\theta^{-1} = p(-k)$$
where $\theta^2 = -I$.
- Denote by $u_p(k)$ the unitary matrix $I - 2p(k)$. In two or three dimensions, the map $\mathbb{T}^d \ni k \;\longmapsto\; u_p(k) \in U(N)$ is then equivariant, i.e. $\Theta(u_p(k)) = u_p(-k)$.

Theorem. 1. For $d = 2$, $\sqrt{Hol_{\mathcal{G}}(u_p)} = (-1)^{KM}$ where $KM \in \mathbb{Z}_2$ is the Fu-Kane-Mele [12, 13] invariant of the time-reversal symmetric $2d$ topological insulators.
2. For $d = 3$, $\mathcal{K}(u_p) = (-1)^{KM^s}$ where $KM^s \in \mathbb{Z}_2$ is the *strong* Fu-Kane-Mele invariant [14] of the time-reversal symmetric $3d$ topological insulators.

Remark. 1. One has a relation between the strong and weak invariants: $KM^s = KM|_{\mathbb{T}^2_0} + KM|_{\mathbb{T}^2_\pi}$.
2. The KM and KM^s invariants count modulo 2 the massless modes carrying edge currents on half-infinite lattice (the bulk-edge correspondence).

VIII. APPLICATION TO FLOQUET SYSTEMS

- Floquet systems are described by lattice Hamiltonians periodically depending on time that, after the discrete Fourier-Bloch transformation, give rise to a map
$$\mathbb{R} \times \mathbb{T}^d \ni (t,k) \;\longmapsto\; h(t,k) = h(t+2\pi,k) \;\in\; End(\mathbb{C}^N)$$
(we fixed for convenience the period of temporal driving to 2π).
- The evolution of such systems is described by the unitary matrices $u(t,k)$ such that
$$\mathrm{i}\partial_t u(t,k) = h(t,k)\,u(t,k), \qquad u(0,k) = I, \qquad u(t+2\pi,k) = u(t,k)\,u(2\pi,k).$$
- Floquet theory is based on the diagonalization of the unitary matrices $u(2\pi,k)$ with eigenvalues $\mathrm{e}^{-\mathrm{i}e_n(k)}$ where $e_n(k)$ are called the (band) "quasienergies".
- Suppose that $\epsilon \in [-2\pi, 0[$ is such that $\mathrm{e}^{-\mathrm{i}\epsilon} \notin spec(u(2\pi,k))$ (i.e. ϵ is in the quasienergy gsp) for all k. Then $h_\epsilon(k) \equiv h_\epsilon(u(2\pi,k)) = \frac{\mathrm{i}}{2\pi}\ln_{-\epsilon}(u(2\pi,k))$ is well defined and
$$v_\epsilon(t,k) = u(t,k)\,\mathrm{e}^{-\mathrm{i}th_\epsilon(k)} = v_\epsilon(t+2\pi,k)$$
may be viewed as a periodized evolution.
- For $\epsilon \le \epsilon'$,
$$h_{\epsilon'}(k) - h_\epsilon(k) = p_{\epsilon,\epsilon'}(u(2\pi,k)) \equiv p_{\epsilon,\epsilon'}(k)$$
where $p_{\epsilon,\epsilon'}(k)$ is the spectral projector of $u(2\pi,k)$ on quasienergies $\epsilon < e_n(k) < \epsilon'$
- For the fermionic ($\theta^2 = -I$) time-reversal symmetric Floquet systems with $\theta h(t,k)\theta^{-1} = h(-t,-k)$,
$$\Theta(v_\epsilon(t,k)) = v_\epsilon(-t,-k) \qquad \text{and} \qquad \theta p_{\epsilon,\epsilon'}(k)\theta^{-1} = p_{\epsilon,\epsilon'}(-k)$$
for $\epsilon \le \epsilon'$.

- In particular, in $2d$ one may consider the Kane-Mele invariants $KM_{\epsilon,\epsilon'} \in \mathbb{Z}_2$ of the quasienergy bands between ϵ and ϵ' given by the relation

$$(-1)^{KM_{\epsilon,\epsilon'}} = \sqrt{Hol_{\mathcal{G}}(u_{p_{\epsilon,\epsilon'}})}$$

where $u_{p_{\epsilon,\epsilon'}}(k) = I - 2p_{\epsilon,\epsilon'}(k)$.

Definition. In $2d$ take $R = \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{T}^2 = \mathbb{T}^3$ with the *orientation-reversing* involution $\rho(t,k) = (-t,-k)$. Then $v_\epsilon : (R,\rho) \longrightarrow (U(N),\Theta)$ and we defined [15, 16] the additional dynamical topological invariants $K_\epsilon \in \mathbb{Z}_2$ of the gapped time-reversal symmetric Floquet system by the relation

$$(-1)^{K_\epsilon} = \mathcal{K}(v_\epsilon).$$

Proposition. The above invariants that dependend on the qusienergy gap ϵ are related by the identity

$$K_{\epsilon'} - K_\epsilon = KM_{\epsilon,\epsilon'}.$$

Remark. The invariants K_ϵ are the counterparts for time-reversal symmetric gapped Floquet systems of the dynamical invariants for such systems without time-reversal symmetry introduced in [17].

- Similarly in $3d$ we may define the *strong* Fu-Kane-Mele invariants $KM^s_{\epsilon,\epsilon'} \in \mathbb{Z}_2$ of the quasienergy bands between ϵ and ϵ' by

$$(-1)^{KM^s_{\epsilon,\epsilon'}} = \mathcal{K}(u_{p_{\epsilon,\epsilon'}}).$$

Definition. In $3d$ take $R = \mathbb{T}^3$ with the *orientation-reversing* involution $\rho(k) = -k$. Then $v_\epsilon|_{t=\pi} : (R,\rho) \longrightarrow (U(N),\Theta)$ and we defined the additional dynamical topological invariants $K^s_\epsilon \in \mathbb{Z}_2$ of the time-reversal symmetric gapped Floquet system by the relation

$$(-1)^{K^s_\epsilon} = \mathcal{K}(v_\epsilon|_{t=\pi}).$$

Proposition. 1. (Relation to the strong Kane-Mele invariant) $K^s_{\epsilon'} - K^s_\epsilon = KM^s_{\epsilon,\epsilon'}$.
2. (Relation to weak invariants) $KM^s_\epsilon = KM_\epsilon|_{\mathbb{T}^2_0} + KM_\epsilon|_{\mathbb{T}^2_\pi}$.

Remark. The indices K_ϵ and KM^s_ϵ should count the parity of the massless modes of the one-period evolution operator that carry edge currents on the half-space lattice system and appear in the bulk spectral gap around quasienergy ϵ.

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