

Higgs bundles and fundamental group schemes

Ugo Bruzzo

Scuola Internazionale Superiore di Studi Avanzati (Trieste)
Istituto Nazionale di Fisica Nucleare

Topological Matter, Strings, K-theory and related areas

Adelaide, September 30, 2016

joint work with I. Biswas and S. Gurjar (arXiv:1607.07207 [math.AG])
(but also D. Hernández Ruipérez, B. Graña Otero, V. Lanza, A. Lo Giudice, ...)

Higgs geometry

X a smooth projective variety

A Higgs bundle \mathfrak{E} is a pair (E, ϕ) , where E is a vector bundle, and

$$\phi: E \rightarrow E \otimes \Omega_X^1$$

(the Higgs field) is a morphism such that $\phi \wedge \phi = 0$, where

$$\phi \wedge \phi: E \rightarrow E \otimes \Omega_X^2$$

The existence of properly semistable Higgs bundles on a variety (or Higgs bundles satisfying some other specific property) is a geometric feature of the variety

(cf. the derived category of coherent sheaves on X)

The classical theory

X a smooth projective variety over a field \mathbb{k} of characteristic zero,
 $H = c_1(\mathcal{O}_X(1))$

A line bundle L on X is **numerically effective** (nef) if for every morphism $f: C \rightarrow X$, where C is a smooth projective irreducible curve, one has

$$\deg f^*L = \int_C f^*c_1(L) \geq 0.$$

If E is a vector bundle on X , the projectivization $\mathbb{P}E$ carries a “relative hyperplane bundle” $\mathcal{O}_{\mathbb{P}E}(1)$. The vector bundle E is said to be nef if $\mathcal{O}_{\mathbb{P}E}(1)$ is.

Definition of $\mathbb{P}E$:

$$E \rightsquigarrow \mathrm{GL}(E) \rightsquigarrow \mathrm{PGL}(E) \rightsquigarrow \mathbb{P}E$$

E is said to be **numerically flat** (nflat) if both E and E^* are nef.

Theorem (Demailly-Peternell-Schneider 1994)

A vector bundle E is nflat if and only if it admits a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

whose quotients E_k/E_{k-1} are Hermitian flat vector bundles (i.e., they are given by representations $\pi_1(X) \rightarrow U(r_k)$).

Consequence: all Chern classes of an nflat bundle vanish (as E and $\bigoplus_k E_k/E_{k-1}$ have the same Chern classes)

Slope of a coherent sheaf \mathcal{F} of positive rank:

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\text{rk } \mathcal{F}} \quad \left(= \frac{1}{\text{rk } \mathcal{F}} \int_X c_1(\det \mathcal{F}) \wedge \omega^{n-1} \right)$$

A torsion-free coherent sheaf \mathcal{F} (e.g., a bundle) is (semi)stable if

$$\mu(\mathcal{E}) (\leq) < \mu(\mathcal{F})$$

for all proper subsheaves \mathcal{E} of \mathcal{F} .

Definition

A vector bundle E on X is *curve semistable* if for all morphisms $f: C \rightarrow X$ (where C is a smooth irreducible projective curve)^a the pullback bundle f^*E is semistable.

^aover \mathbb{C} a.k.a. a Riemann surface

Theorem (B - Hernández Ruipérez; Nakayama)

Let E be a vector bundle on X . The following conditions are equivalent:

- E is curve semistable;
- E is semistable and $\Delta(E) = 0$.

$$\Delta(E) = c_2(E) - \frac{r-1}{2r}c_1(E)^2 \in H^4(X, \mathbb{Q})$$

Simple fact: A bundle E with vanishing first Chern class is curve semistable if and only if it is nflat.

Definition

A neutral Tannakian category over a field \mathbb{k} is a rigid Abelian \mathbb{k} -linear tensor category \mathcal{T} together with an exact faithful \mathbb{k} -linear tensor functor $\omega: \mathcal{T} \rightarrow \mathbf{Vect}_{\mathbb{k}}$, called the fiber functor.

“Rigid” means that

- Hom and \otimes satisfy a distributive property (over finite families);
- all objects in \mathcal{T} are reflexive.

Archetypical Tannakian category: $\text{Rep}(G)$ for G an affine group scheme over \mathbb{k}

$$\omega(\rho, V) = V \quad \text{if} \quad \rho: G \rightarrow \text{Aut}(V).$$

Theorem (Tannaka duality)

For every neutral Tannakian category (\mathfrak{T}, ω) there is an affine group scheme G such that $\mathfrak{T} \simeq \text{Rep}(G)$.

Actually $G = \text{Aut}^{\otimes}(\omega)$.

Nori fundamental group

A vector bundle E over a scheme X is **essentially finite** if there exists a principal bundle $\pi: P \rightarrow X$, with a finite structure group, such that π^*E is trivial.

Essentially finite vector bundles make up a neutral Tannakian category (the fiber functor maps E to the fiber over a fixed point $x \in X$). The affine group scheme representing this Tannakian category is the **Nori fundamental group scheme** $\pi_1^N(X, x)$.

Relation with the usual fundamental group:

$$\pi_1^N(X, x) \twoheadrightarrow \pi_1^{\text{ét}}(X, x)$$

$$\pi_1^{\text{ét}}(X, x) = \varprojlim_i \text{Aut}((X_i, x_i))$$

$((X_i, x_i)$ pointed Galois coverings of (X, x))

$\pi_1^{\text{ét}}(X, x)$ is a pro-finite completion of $\pi_1(X, x)$

$$G^\wedge = \varprojlim_i G/N_i$$

$(N_i$ normal subgroups of finite index)

The S -fundamental group

- If $f: E \rightarrow F$ is a morphism of nflat bundles, $\ker(f)$ and $\operatorname{im}(f)$ are both nflat bundles, and $\operatorname{rk}(f)$ is constant
- the tensor product of nflat bundles is nflat

Let $\mathbf{NF}(X)$ be the category of nflat bundles on X , fix $x \in X$ and define a functor

$$\varpi_x: \mathbf{NF}(X) \rightarrow \mathbf{Vect}, \quad E \mapsto E_x$$

Theorem (Langer)

$(\mathbf{NF}(X), \varpi_x)$ is a neutral Tannakian category

$\rightsquigarrow \pi_1^S(X, x)$ the S -fundamental group scheme (Langer)

Over \mathbb{C} (and modulo a technical condition) $\pi_1^S(X, x)$ coincides with **Simpson's universal complex fundamental group**, which carries information on all finite-dimensional representations of the topological fundamental group

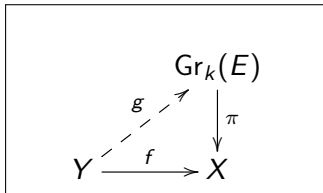
Grassmann bundle

E vector bundle on X , $\text{Gr}_k(E) \xrightarrow{\pi} X$ its k -th Grassmann bundle (bundle of k -planes in E)

$$Y \xrightarrow{f} X, \quad f^*E \rightarrow F \rightarrow 0$$

$$F = g^*Q_k$$

\uparrow
rank k universal quotient bundle



$$0 \rightarrow S_k \rightarrow \pi^*E \rightarrow Q_k \rightarrow 0$$

If E is nef, all universal quotients bundles are nef (and viceversa, as $\text{Gr}_1(E) = \mathbb{P}E$ and $Q_1 = \mathcal{O}_{\mathbb{P}E}(1)$) \longrightarrow use this to get a definition of “numerical effectiveness” for Higgs bundles

Higgs Grassmannians

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S_k & \longrightarrow & \pi^* E & \longrightarrow & Q_k \longrightarrow 0 \\
 & & \downarrow & & \downarrow \pi^* \phi & & \downarrow \\
 0 & \longrightarrow & S_k \otimes \Omega_{\text{Gr}_k(E)}^1 & \longrightarrow & \pi^* E \otimes \Omega_{\text{Gr}_k(E)}^1 & \longrightarrow & Q_k \otimes \Omega_{\text{Gr}_k(E)}^1 \longrightarrow 0
 \end{array}$$

(morphism of exact sequences of vector bundles on $\text{Gr}_k(E)$)

Definition

The k -th Higgs Grassmannian of the Higgs bundle $\mathcal{E} = (E, \phi)$ is the closed subscheme

$$\mathfrak{Gr}_k(\mathcal{E}) \subset \text{Gr}_k(E)$$

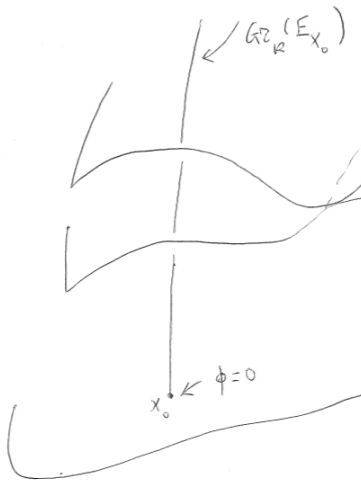
where the the composition of the above blue arrows vanishes

$\mathcal{G}r_k(\mathbb{C})$ is a very nasty scheme (it can be singular, nonreduced, non-equidimensional, ...)

$\mathcal{G}r_k(\mathcal{E})$ is a very nasty
scheme (it can be
singular, nonreduced,
non-equidimensional, ...)
but we shall be brave
and will work
with it.



$\mathcal{G}r_k(\mathcal{E})$ is a very nasty scheme (it can be singular, nonreduced, non-equidimensional, ...) but we shall be brave and will work with it.



Definition

If $\mathcal{E} = (E, \phi)$ is a Higgs bundle, a quotient Q of E is a Higgs quotient if the corresponding kernel F is ϕ -invariant, $\phi(F) \subset F \otimes \Omega_X^1$.

Fact: Q is a Higgs quotient if and only if the corresponding section σ_Q of $\mathrm{Gr}_k(E)$ takes values in $\mathcal{G}\tau_k(\mathcal{E}) \subset \mathrm{Gr}_k(E)$.

The universal quotient Q_k restricted to $\mathcal{G}\tau_k(\mathcal{E})$ has a natural Higgs field Φ_k , so that we have a Higgs bundle $\Omega_{k, \mathcal{E}} = (Q_k|_{\mathcal{G}\tau_k(\mathcal{E})}, \Phi_k)$ on $\mathcal{G}\tau_k(\mathcal{E})$, and

$$\Omega = (Q, \phi_Q) = \sigma_Q^* \Omega_{k, \mathcal{E}}.$$

Numerically flat Higgs bundles

Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle.

Definition

If $\text{rk } E = 1$, \mathfrak{E} is H -nef if E is nef in the usual sense.

If $\text{rk } E > 1$, \mathfrak{E} is H -nef if

- $\det E$ is nef in the usual sense*
- Every universal quotient Higgs bundle \mathfrak{Q}_k is H -nef.*

\mathfrak{E} is H -nflat if both \mathfrak{E} and \mathfrak{E}^ are H -nef.*

A notion of semistability of Higgs bundles is introduced as for usual bundles, but checking the inequality between the slopes only for Higgs-invariant subsheaves.

Definition

A Higgs bundle \mathcal{E} on X is curve semistable if for all morphisms $f: C \rightarrow X$ the pullback Higgs bundle $f^\mathcal{E}$ is semistable.*

Theorem (B - Hernández Ruipérez - Graña Otero, 2004, 2006)

If \mathcal{E} is semistable (with respect to some polarization) and $\Delta(E) = 0$, then it is curve semistable.

The opposite implication is a conjecture (B – Graña Otero, 2010).

Proposition

The previous conjecture is equivalent to the following statement: all Chern classes of an H-nflat Higgs bundle vanish.

Work done with A. Lo Giudice and V. Lanza shows that the conjecture holds for some classes of varieties, e.g.

- varieties whose tangent bundle is nef
- K3 surfaces (partial results)

and for some close relatives.

The challenge is to prove the conjecture for varieties of general type (canonical bundle is ample).

The fundamental Higgs scheme

Some properties of H-nflat Higgs bundles:

- a morphism of H-nflat Higgs bundles has constant rank;
- the kernel and cokernel of a morphism of H-nflat Higgs bundles are H-nflat Higgs bundles;
- the tensor product of two H-nflat Higgs bundles is H-nflat.

Theorem (Biswas - B - Gurjar, 2016)

The category $\mathbf{HNF}(X)$ of H-nflat Higgs bundles on X is a neutral Tannakian category

We denote by $\pi_1^H(X, x)$ the group scheme which represents it (**the Higgs fundamental group scheme of X**).

Since $\mathbf{NF}(X) \subset \mathbf{HNF}(X)$, there is a surjection

$$\pi_1^H(X, x) \twoheadrightarrow \pi_1^S(X, x)$$

HFGS vs. BGO's conjecture

- if $\pi_1^H(X, x) = \{e\}$ then $\mathbf{HNF}(X) = \mathbf{Vect}$, so that all H-nflat Higgs bundles are trivial, and the conjecture holds for X ;
- if $\pi_1^H(X, x) = \pi_1^S(X, x)$ then $\mathbf{HNF}(X) = \mathbf{NF}(X)$, and all H-nflat Higgs bundles are nflat, so that the conjecture holds for X ;

Moreover there is a morphism

$$\pi_1^H(X \times Y, (x, y)) \rightarrow \pi_1^H(X, x) \times \pi_1^H(Y, y) \quad (*)$$

We do not know if this is an isomorphism. This is related to the conjecture: indeed, if $(*)$ is an isomorphism (at least for products of curves), then the conjecture holds for products of curves.