# Distribution of eigenvalues of families of unitary operators

Andrew Hassell

Mathematical Sciences Institute
Australian National University

Adelaide, October 2015

Suppose that we have a family of unitary operators $U(h)$, parametrized by $h > 0$. As unitary operators, the spectrum lies on the unit circle. Let us make the assumption that the spectrum is discrete away from the point 1 on the unit circle. Then we can count the number of eigenvalues in any interval $I$ of the circle away from 1.

When the family $U(h)$ arises from a geometric/analytic setting, one might expect to have an asymptotic for the number of eigenvalues of $U(h)$ in $I$, as $h \to 0$.

I will describe several examples where this has been achieved.

# Semiclassical potential scattering

The first example is semiclassical scattering by a potential function. The Hamiltonian is

$$H = h^2 \Delta + V(x)$$

on $\mathbb{R}^n$, where $V(x)$ is a smooth potential function. We fix an energy level $E > 0$ and consider solutions to the equation

$$H\psi = E\psi.$$

I will assume one of the following two conditions:

- $V$ compactly supported and $C^\infty$;
- $V$ is a classical symbol at infinity with order $-\alpha$, $\alpha > n$. In particular,

$$V(x) \sim V_0(x/|x|)|x|^{-\alpha} \text{ as } |x| \to \infty, \quad V_0 \neq 0.$$

# Scattering matrix

The scattering matrix $S_h(E)$ may be defined through the asymptotics of generalized eigenfunctions $\psi$ of $H$. As is well known, for each $f \in C^\infty(S^{n-1})$ there is a unique generalized eigenfunction $\psi_f$ with asymptotics

$$\psi_f(x) = r^{-(n-1)/2}\Big(e^{-i\sqrt{E}r/h}f(\hat{x}) + e^{i(n-1)\pi/2}e^{i\sqrt{E}r/h}g(-\hat{x})\Big) + O(r^{-(n+1)/2}), \quad r = |x|,\ \hat{x} = \frac{x}{|x|}, \quad g \in C^\infty(S^{n-1}).$$

The map $f \mapsto g$ is the scattering matrix $S_h(E)$. It extends to a unitary transformation on $L^2(S^{n-1})$. It is normalized so that the scattering matrix for the zero potential is the identity.

# Properties of the scattering matrix

- It is unitary, so its spectrum lies on the unit circle.

- For compactly supported potentials, $S_h(E) = \mathrm{Id} + A_h(E)$ where $A_h(E)$ is a smoothing operator, in particular compact.
- For potentials in $S_{cl}^{-\alpha}$, $\alpha > 1$, $A_h(E)$ is a pseudodifferential operator of order $-\alpha + 1$, for fixed $h$, and again compact. For $\alpha > n$, then $A_h(E)$ is trace class.
- In either case, the spectrum of $S_h(E)$ is discrete away from 1 on the unit circle.

- Hence we can count the number of eigenvalues of the scattering matrix in a closed interval $I$ of the unit circle not containing 1. What are the asymptotics as $h \to 0$?

# Semiclassical scattering matrix

The structure of the semiclassical scattering matrix is known due to work of Guillemin, Majda-Ralston, Robert-Tamura, Alexandrova, Hassell-Wunsch. If the potential is nontrapping at energy $E$, then the scattering matrix is a semiclassical FIO, associated to a canonical transformation called the **reduced scattering transformation** determined by the **classical** motion for the classical Hamiltonian $|\xi|^2 + V(x)$ at energy $E$.

For a compactly supported potential:

Given $(y', \eta') \in T^* S^{n-1}$, there is a unique trajectory of the form

$$x(t) = ty' + \eta', \quad t \to -\infty.$$

Here we think of $y'$ as a unit vector in $\mathbb{R}^n$ and $\eta'$ as a vector orthogonal to $y$. For $t$ large, assuming that the trajectory is not trapped, then we have

$$x(t) = ty + \eta, \quad |y| = 1, \ \eta \perp y.$$

The reduced scattering transformation is $(y', \eta') \to (y, \eta)$. It is a symplectic map on $T^* S^{n-1}$.

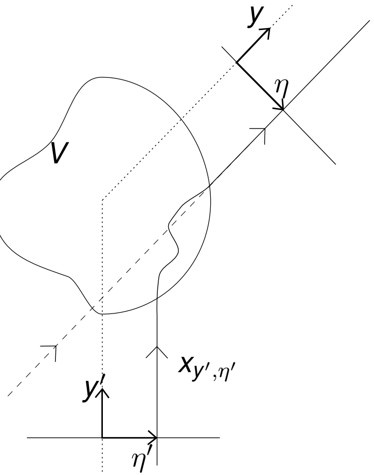


Figure: The scattering relation. Here $(y', \eta')$ lies in the interaction region $\mathcal{I}$. The long-dashed line depicts how the outgoing data $(y, \eta)$ is also in $\mathcal{I}$.

# Central potentials

For a **central** potential, the dynamics is essentially two-dimensional. Working on $T^*S^1$ with coordinates $(\theta, \eta)$, the scattering transformation takes the form

$$(\theta, \eta) \mapsto (\theta + \Sigma(\eta), \eta)$$

due to rotational invariance and conservation of angular momentum. Here $\Sigma(\eta)$ is (manifestly) the scattering angle, i.e. the difference between the final and initial angle.

Suppose that $V$ is both compactly supported and central, and $n = 2$. The scattering matrix is then a map on $L^2(S^1)$, depending on $h$. So its integral kernel is a function of two angle variables $\theta, \theta'$.

Rotational symmetry implies that $S_h(E)(\theta, \theta')$ depends only on $\theta - \theta'$, and the angular momentum is preserved. It follows that the scattering matrix takes the form

$$(2\pi h)^{-1} \int e^{i\big((\theta-\theta')\eta + G(\eta)\big)/h} a(\theta - \theta', \eta, h)\, d\eta \tag{1}$$

where $-G'(\eta) = \Sigma(\eta)$ is the scattering angle.

# More general potentials

If the potential is not central, the scattering matrix is a more general Fourier integral operator of the form

$$(2\pi h)^{-1} \int e^{i\big((\theta-\theta')\eta + G(\theta,\eta)\big)/h} a(\theta,\theta',\eta,h)\, d\eta. \qquad (2)$$

In either case, the canonical relation, i.e. the set

$$\big\{(\theta,\theta', d_\theta\Phi, -d_{\theta'}\Phi) \mid d_\eta\Phi = 0\big\}$$

is the graph of the reduced scattering transformation. This is the **classical-quantum correspondence** in this setting.

To state our result, let $V$ be a compactly supported potential. Define the **interaction region** $\mathcal{I} \subset T^*S^{n-1}$ to be the set of all $(y, \eta) \in T^*S^{n-1}$ such that the corresponding classical trajectory meets the support of $V$. It is not hard to see that this is **invariant** under the reduced scattering transformation. Also, because $V$ is compactly supported, $\mathcal{I}$ is compact, hence has finite measure.

We also let $N_h(\phi_0, \phi_1)$ denote the number of eigenvalues $e^{i\beta}$ of $S_h(E)$ such that $\phi_0 \le \beta \le \phi_1$ (mod $2\pi$), counted with multiplicity.

**Theorem (Gell-Redman, H., Zelditch)**

*Let $V$ be a compactly supported potential. Assume that*
- *$E$ is a nontrapping energy for $|\xi|^2 + V$ and*
- *On $\mathcal{I}$, the fixed point set of every power of the reduced scattering transformation has measure zero.*

*Then for all $0 < \phi_0 < \phi_1 < 2\pi$, we have*

$$\frac{N_h(\phi_0, \phi_1)}{(2\pi h)^{-(n-1)}} \to \frac{\phi_1 - \phi_0}{2\pi} \operatorname{vol}(\mathcal{I}) \text{ as } h \to 0. \tag{3}$$

- Previously, Datchev, Gell-Redman, H. and Humphries proved this result (with slightly different assumptions) for central, compactly supported potentials.

- Compare with the usual Weyl asymptotics for a positive self-adjoint elliptic semiclassical operator $A \in \Psi_h^0(M)$ with principal symbol $a(x, \xi)$ on a closed manifold $M$ of dim. $d$. If $N_h(E)$ is the number of eigenvalues of $A$ less than $E$, then

$$\frac{N_h(E)}{(2\pi h)^{-d}} \to \mathrm{vol}\{(x, \xi) \in T^*M \mid a(x, \xi) \leq E\}.$$

- We can express the result as follows: if we define the measure

$$\mu_h = (2\pi h)^{(n-1)} \sum_{e^{i\beta} \in \operatorname{spec} S_h(E)} \delta_{e^{i\beta}},$$

then $\mu_h$ tends weak-$*$ to $\mathrm{vol}(\mathcal{I})\, d\theta/2\pi$ on $I \subset S^1 \setminus \{1\}$.

# Potentials with polynomial decay

Our second result is about the scattering matrix for potentials $V \in S^{-\alpha}(\mathbb{R}^n)$ that satisfy

$$V(x) = r^{-\alpha} a_0(\hat{x}) + O(r^{-(\alpha+\epsilon)}), \quad x \to \infty,$$

where $a_0$ is smooth and strictly positive.

- Not clear how our first result might generalize to potentials with noncompact support. However, we argue heuristically:

- For each $h$, there is an effective interaction region of radius $h^{-\beta}$ outside of which $V$ is semiclassically negligible.
- $\#$ eigenvalues of $S_h(E)$ essentially different from 1 should be about $h^{-(n-1)} \times h^{-\beta(n-1)}$, where $h^{-\beta(n-1)}$ is the approx. volume of phase space meeting this effective support.
- So the number of eigenvalues that lie in a given interval away from 1 should grow as $h^{-(n-1)(1+\beta)}$.

It turns out that $\beta = 1/(\alpha - 1)$. So we define

$$\mu_h = (2\pi h)^{(n-1)\alpha/(\alpha-1)} \sum_{e^{i\beta} \in \mathrm{spec} S_h(E)} \delta_{e^{i\beta}}.$$

Let $\mu_{a_1,a_2}$ denote the measure on $S^1$ that is the pushforward of the measure on $\mathbb{R}$

$$\begin{cases} a_1 x^{-\gamma}\, dx, \ x > 0 \\ a_2 |x|^{-\gamma} dx, \ x < 0 \end{cases} \qquad \gamma = 1 + \frac{n-1}{\alpha - 1}$$

by the map $x \mapsto e^{ix}$. Notice $1 < \gamma < 2$, so $x^{-\gamma} dx$ has finite mass as $x \to \infty$. Therefore $\mu_{a_1,a_2}$ is finite except for an infinite accumulation of mass near 1.

## Theorem (Gell-Redman, H.)

*Suppose that $V \in S_{cl}^{-\alpha}(\mathbb{R}^n)$ is as above, and is nontrapping at energy $E$. Then for some constants $a_1, a_2$ depending on the leading asymptotic of $V$, and all $0 < \phi_0 < \phi_1 < 2\pi$, we have*

$$\frac{N(\phi_0, \phi_1)}{(2\pi h)^{-(n-1)\alpha/(\alpha-1)}} \to c \int_{\phi_0}^{\phi_1} d\mu_{a_1,a_2} \text{ as } h \to 0. \tag{4}$$

*Equivalently,*

$$\mu_h \to \mu_{a_1,a_2} \text{ in the weak-}* \text{ topology on } I, \text{ as } h \to 0.$$

# Dirichlet-to-Neumann operator

Our third result comes from compact Riemannian manifolds $M$. Let $\Delta$ be the positive Laplacian on $M$. Using the Laplacian, we define a unitary transformation $C(h)$ as follows. Given $f \in C^\infty(\partial M)$, we solve the equation

$$(h^2\Delta - 1)u = 0 \text{ on } M, \quad hd_n u - iu = f \text{ at } \partial M.$$

We then define

$$C(h)f = hd_n u + iu\Big|_{\partial M}.$$

This is related to the Dirichlet-to-Neumann operator $R(h)$. This operator sends $f \in C^\infty(\partial M)$ to $d_n u\big|_{\partial M}$ where $u$ is the solution to

$$(h^2\Delta - 1)u = 0, \quad u\Big|_{\partial M} = f.$$

It is easy to check that $C(h)$ is the Cayley transform of $hR(h)$:

$$C(h) = (hR(h) + i)^{-1}(hR(h) - i).$$

However, $R(h)$ is undefined whenever $h^{-2}$ is a Dirichlet eigenvalue of the Laplacian, while $C(h)$ is well defined for all $h > 0$. So $C(h)$ seems to be a more natural family of operators than $R(h)$.

## Theorem (H., Ivrii)

*Suppose that the set of periodic geodesics on $M$ has measure zero (as a subset of $T^*M$). Define a measure*

$$\mu_h = h^{(n-1)} \sum_{e^{i\beta} \in \operatorname{spec} C(h)} \delta_{e^{i\beta}}.$$

*Then, the measure $\mu_h$ converges weak-$*$ to a measure $m(\theta)d\theta$, where $m$ is smooth for $\theta \in [0, 2\pi)$ and tends to infinity as $\theta \to 2\pi$. In three dimensions we have*

$$m(\theta) = \frac{\mathrm{Area}(\partial M)}{4\pi} \Big( 2 - (\theta - \sin\theta) \frac{\cos(\theta/2)}{\sin^3(\theta/2)} \Big). \qquad (5)$$

*In other dimensions, we have an integral expression for $m(\theta)$.*

# Proofs

The idea of the proof comes from an earlier paper of Zelditch on quantized contact transformations. These are given by families of unitary matrices with the dimension $N \to \infty$.

Given a sequence $U_N$ of $N$-dimensional unitary matrices, we form a sequence of measures on the unit circle,

$$\mu_N = \frac{1}{N} \sum_{e^{i\beta} \in \mathrm{spec}\, U_N} \delta_{e^{i\beta}}.$$

The measures $\mu_N$ **equidistribute**, i.e. tend to $d\theta/2\pi$ weak-$*$,

$$\text{if and only if} \quad \lim_{N\to\infty} \int_{S^1} f(z) d\mu_N(z) \to \int_{S^1} f(z) \frac{d\theta}{2\pi}$$

for all continuous $f$ (using $z = e^{i\theta}$ as a coordinate on $S^1$).

It is enough to show for $f = z^m$, for $m \in \mathbb{Z}$. But

$$\int_{S^1} z^m d\mu_N(z) = \frac{1}{N} \operatorname{tr}(U_N^m),$$

so a sufficient condition for equidistribution is that

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(U_N)^m = 0 \text{ for all } m \neq 0 \in \mathbb{Z}.$$

# Proof of the first result

We use the method above, but we must deal with the fact that the scattering matrix has infinitely many eigenvalues for every $h$. We define the measure

$$\mu_h = (2\pi h)^{n-1} \sum_{e^{i\beta} \in \operatorname{spec} S_h(E)} \delta_{e^{i\beta}}. \tag{6}$$

The theorem asserts that $\mu_h \to (\operatorname{vol} \mathcal{I}) d\theta/2\pi$ weak-$*$ away from the point 1. That is, we need to show

$$\int_{S^1} f d\mu_h \to \operatorname{vol} \mathcal{I} \int_{S^1} f(e^{i\theta}) \frac{d\theta}{2\pi} \tag{7}$$

for each continuous $f(z)$ **supported away from** $z = 1$**.**

To deal with the accumulation of delta-measures at $z = 1$, we define a weighted Banach space

$$C^0_w(S^1) := \{f \in C^0(S^1) : f = (z-1)g,\ g \text{ continuous}\}. \quad (8)$$

with norm

$$\|f\|_w = \sup_{|z|=1, z\neq 1} \left| \frac{f(z)}{z-1} \right|. \quad (9)$$

Then it turns out that the measure $\mu_h$ is in the dual space of this Banach space for all $h > 0$. (This is equivalent to $S_h(E) - \mathrm{Id}$ being trace class.)

We have

**Lemma**

*The set*

$$\{p \in C^0_w(S^1) \mid p \text{ polynomial}, p(1) = 0\}$$

*is dense in* $C^0_w(S^1)$ *(with respect to the* $C^0_w(S^1)$ *norm).*

**Lemma**

*For every* $f \in C^0_w(S^1)$*, we have an estimate*

$$\langle \mu_h, f \rangle \leq C \|f\|_{C^0_w(S^1)}$$

*with* $C$ *independent of* $h$*.*

As a consequence of these two lemmas, to show (7), it suffices to show that for each $m \neq 0 \in \mathbb{Z}$, we have

$$\langle \mu_h, 1 - z^m \rangle \to \text{vol}(\mathcal{I}) \text{ as } h \to 0. \tag{10}$$

But the LHS is precisely $(2\pi h)^{n-1}$ times the **trace** of $\text{Id} - S_h(E)^m$.

In the case of a compactly supported potential, $S_h(E)$ is an FIO which is equal to the identity (microlocally) outside the interaction region in $T^* S^{n-1}$. That is, $S_h(E) - \mathrm{Id}$ has **compact microsupport**.

The trace of a semiclassical FIO $F(h)$ of order zero and compact microsupport is given, in dim. $n-1$, by

$$\operatorname{tr} F(h) = (2\pi h)^{-(n-1)} \int_{\mathrm{Fix}(C)} e^{i\tau(F)/h} \sigma(F) + o(h^{-(n-1)})$$

where $\mathrm{Fix}(C)$ is the fixed point set for the canonical relation of $F$. Here, the difference between Id and $S_h^m(E)$ is $O(h^\infty)$ away from the interaction region $\mathcal{I}$, while on the interaction region, by assumption the canonical relation for $S_h^m(E)$ has zero measure fixed point set. So the only contribution is from the identity restricted to the interaction region, showing (10).

Next consider the proof of (4) for $V$ with polynomial decay. In two dimensions, the proof goes as follows: we know that we can write the scattering matrix in the form

$$(2\pi h)^{-1} \int e^{i\big((\theta-\theta')\eta+G(\theta,\eta)\big)/h} a(\theta,\theta',\eta,h)\, d\eta. \qquad (11)$$

Here, $G(\theta,\eta)$ will be $O(|\eta|^{1-\alpha})$ as $\eta \to \pm\infty$. It turns out that the $m$th power of the scattering matrix takes the form

$$(2\pi h)^{-1} \int e^{i\big((\theta-\theta')\eta+mG(\theta,\eta)+\tilde{G}(\theta,\eta)\big)/h} a(\theta,\theta',\eta,h)\, d\eta,$$

$$\text{where} \quad |\tilde{G}(\theta,\eta)| \le C|\eta|^{2(1-\alpha)},$$

that is, it is smaller at infinity than the leading term $mG$.

We then get an expression for the trace of $\mathrm{Id} - S_h(E)^m$ of the form

$$ch^{-\alpha/\alpha-1}\Big( \int \big(e^{imG(\theta,\eta)/h} - 1\big)\, d\theta d\eta + o(1) \Big).$$

We compute the integral and find that it gives a power of $m$, namely $c(\operatorname{sgn} m)m^{\gamma-1}$.

- This matches the integrals of $\mu_{a_1,a_2}$ against $1 - z^m$, for suitable $a_1, a_2$, which completes the proof.

The first result, on equidistribution, can only occur if there is an infinite atom at 1 (since the total mass of $\mu$ must be infinite). This implies that the measure can be cleanly separated into two parts. Classically this requires that the phase space is also divided cleanly into two parts, which is true in the compactly supported case but not the polynomially decaying case. Equidistribution doesn't even make sense as a possibility in the latter case!

# Proof of the third result

The proof of the third result, (5), on $C(h)$, is completely different. Here we relate the number of eigenvalues of $C(h)$ in an interval $I$ of the unit circle to eigenvalue counting functions for the domain $\Omega$.

Notice that if $f$ is an eigenfunction of $C(h)$ with eigenvalue $e^{i\theta}$, then it is an eigenvalue of the semiclassical Dirichlet-to-Neumann operator $R(h)$ with eigenvalue $a = -\cot(\theta/2)$. Then the corresponding Helmholtz function $u$ is in the domain of the operator $P_a$, where $P_a$ is the operator $h^2\Delta - 1$ with

$$\mathfrak{D}(P_a) = \{u \in H^2(M) : (h\partial_\nu - a)u = 0 \text{ at } \partial M\}. \tag{12}$$

Moreover, $P_a u = 0$.

$P_0 =$ Neumann Laplacian, and "$P_{-\infty} =$ Dirichlet Laplacian." For other $a$ this is a genuinely semiclassical family of operators.

The operator $P_a$ is the self-adjoint operator associated to the quadratic form on $H^1(M)$ given by

$$h^2\|\nabla u\|_M^2 - \|u\|_M^2 - ha\|u\|_{\partial M}^2. \tag{13}$$

Since this quadratic form is monotone in $a$, the operators $P_a$ are monotone (decreasing) in $a$. In particular, the eigenvalues are decreasing in $a$.
Let $N_h^-(a)$ denote the number of negative eigenvalues of $P_a$.
The Birman-Schwinger principle tells us that

$$N_h^-(-\cot(\theta_2/2)) - N_h^-(-\cot(\theta_1/2)) = \#\{\mathrm{spec}(C(h)\cap(\theta_1, \theta_2]\}.$$

The proof is a diagram!

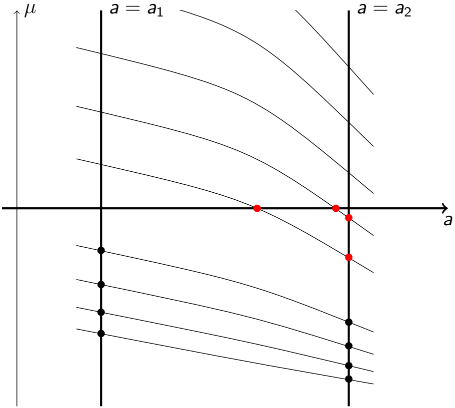


Figure 1: Diagram showing the variation of eigenvalues $\mu(a, h)$ of $P_{a,h}$ as a function of $a$ for fixed $h$. The eigenvalues are strictly decreasing in $a$. Consequently, the number of negative eigenvalues of $P_{a_2,h}$ is equal to the number of negative eigenvalues of $P_{a_1,h}$ together with the number that cross the $a$-axis between $a = a_1$ and $a = a_2$.

Then the result follows from a **two-term expansion** of the counting function for the operator $P_a$. This takes the form

$$N_h^-(a) = (2\pi h)^{-n}\omega_n \operatorname{vol}_n(M) + h^{1-n}\kappa(a)\operatorname{vol}_{n-1}(\partial M) + o(h^{1-n}),$$

where

$$\kappa(a) = (2\pi)^{1-n}\omega_{n-1}\Big( -\frac{1}{2\pi}\int_{-1}^{1}(1-\eta^2)^{(n-1)/2}\frac{a}{a^2+\eta^2}\,d\eta \\ -\frac{1}{4} + H(a)(1+a^2)^{(n-1)/2}\Big) \quad (14)$$

provided that the set of periodic billiard orbits in $M$ has measure zero. Then $m$ in (5) is

$$m(\theta) = \frac{d}{d\theta}\big(\kappa(-\cot(\theta/2))\big).$$