Chern-Simons classes on loop spaces and diffeomorphism groups

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Outline

1. Diffeomorphism groups, Sasakian manifolds and statement of the main results
2. Loop space geometry
   - Natural connections on LM
   - Characteristic classes on TLM
3. Chern-Simons classes on TLM
4. Relating CS classes on TLM to Diff(M)

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- Results on $\pi_i(\mathrm{Diff}(S^n))$ in the stable range $n - i \gg 0$ [Farrell-Hsiang, 1970s]; $\pi_1(\mathrm{Diff}(S^5)) = ?$

Sasakian manifolds

Let (M, ω) be a compact integral Kähler manifold ($\Leftrightarrow M$ is smooth projective algebraic). There is a circle bundle $(S, \nabla_S) \to M$ with connection associated to (M, ω) with $c_1(\Omega_S) = \omega$. For $k \in \mathbb{Z}$, we get (S_k, ∇_k) associated to $k\omega$. Let $\overline{M}_k$ be the total space of S_k.

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Example: $(\mathbb{CP}^n, \omega^{FS})$ has $\overline{\mathbb{CP}}^n_1 \approx S^{2n+1}$ and $\overline{\mathbb{CP}}^n_{\pm k} \approx L_k = S^{2n+1}/\mathbb{Z}_k$.

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$\overline{M}_k$ is a *Sasakian manifold* of dimension $2\ell + 1$ if $\dim_{\mathbb{C}} M = \ell$. ($\overline{M}_k$ has a canonical Riemannian metric $\bar{g}$, a canonical vector field given by unit vertical vectors $\bar{\xi}$, an "almost complex structure" Φ with $\Phi^2 = -I + \bar{\xi}^\sharp \otimes \bar{\xi}$, compatibility of the LC connection with Φ, etc.)

Sasakian manifolds

The geometry of $\overline{M}_k$ is determined by the geometry of M.

Lemma

Let X, Y, Z, W be tangent vectors to $(M, \omega, \langle \cdot, \cdot \rangle)$, and let X^L, etc. be their horizontal lifts to $(\overline{M}_k, \bar{g})$. Then

$$\begin{aligned}
\bar{g}(\bar{R}(X^L, Y^L)Z^L, W^L) &= \langle R(X, Y)Z, W\rangle + k^2[-\langle JY, Z\rangle\langle JX, W\rangle \\
&\quad + \langle JX, Z\rangle\langle JY, W\rangle + 2\langle JX, Y\rangle\langle JZ, W\rangle], \\
\bar{g}(\bar{R}(X^L, Y^L)Z^L, \bar{\xi}) &= 0, \\
\bar{g}(\bar{R}(\bar{\xi}, X^L)Y^L, \bar{\xi}) &= k^2\langle X, Y\rangle.
\end{aligned}$$

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Question: This circle action gives an element in $\pi_1(\mathrm{Diff}(\overline{M}))$ and in fact an element of $\pi_1(\mathrm{Isom}(\overline{M}))$. When is this element nonzero? When does it have infinite order?

Main results

Example: For $\overline{M} = \overline{\mathbb{CP}^n}_1 = S^{2n+1}$, the circle action gives a generator of $\pi_1(\mathrm{Isom}(\overline{M})) = \pi_1(SO(2n+2)) = \mathbb{Z}_2$. This element has order two.

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Theorem

(i) Let (M, ω) be an integral Kähler surface. Then the circle action is an element of infinite order in $\pi_1(\mathrm{Diff}(\overline{M}_k))$ and in $\pi_1(\mathrm{Isom}(\overline{M}))$ for $k \gg 0$.

(ii) Let (M, ω) be an integral Kähler manifold of real dimension 4ℓ. If the signature $\sigma(M) \neq 0$, then the circle action is an element of infinite order in $\pi_1(\mathrm{Diff}(\overline{M}_k))$ for $k \gg 0$.

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Any compact Kähler surface is deformable to an algebraic surface, so Theorem (i) for $\mathrm{Diff}(M)$ applies to all Kähler surfaces.

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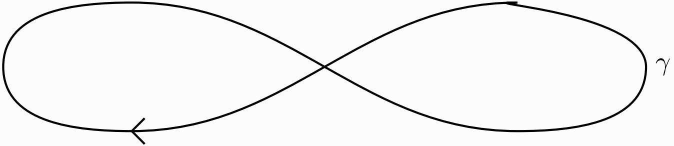

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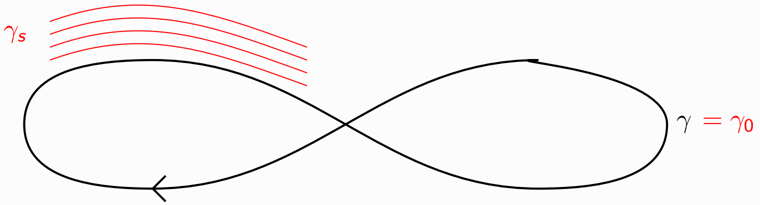

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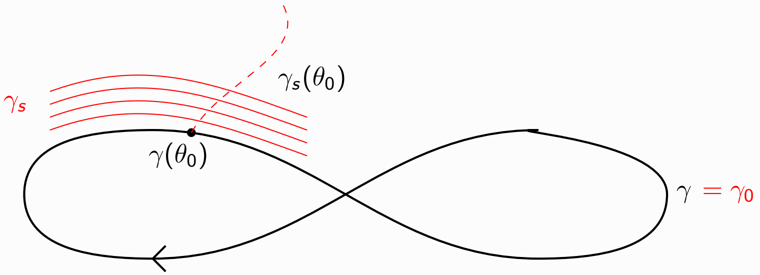

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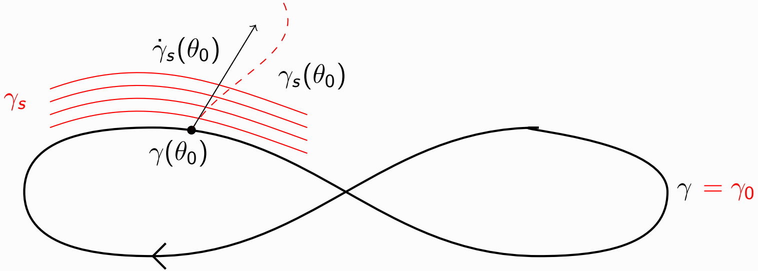

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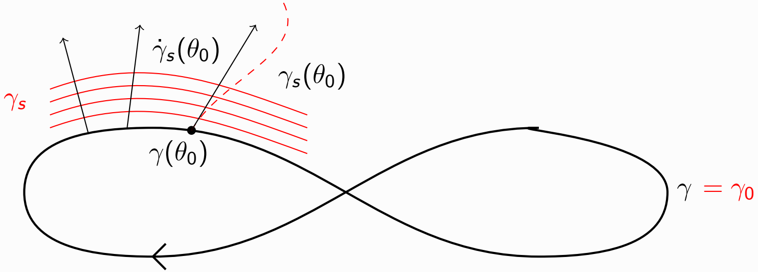

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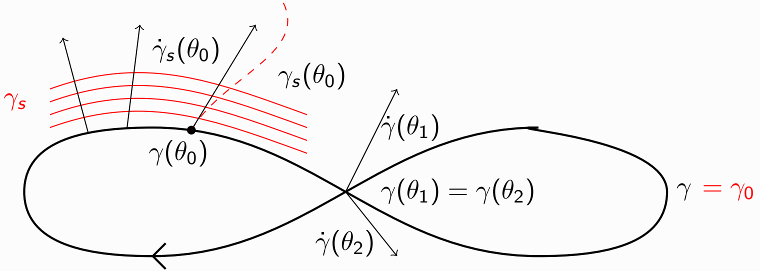

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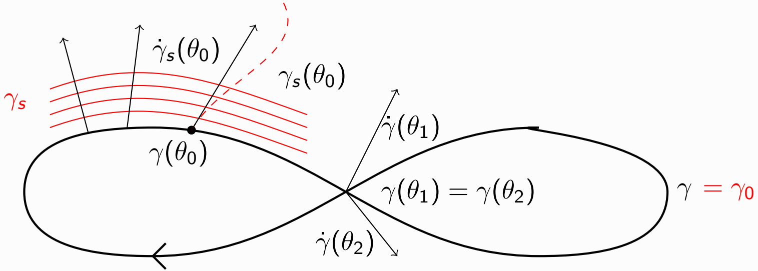

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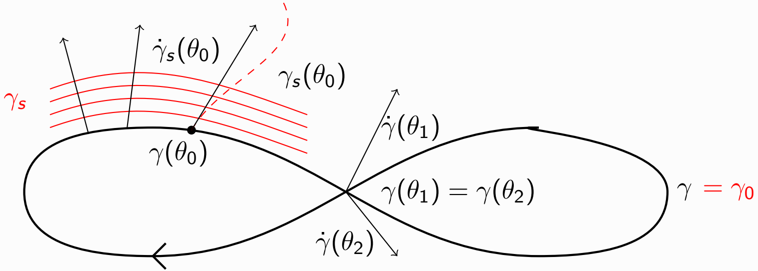

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The structure group of TM is $GL(n, \mathbb{R})$; the structure group of TLM is $\mathcal{G} = Maps(S^1, GL(n, \mathbb{R}))$, the gauge transformations of $S^1 \times \mathbb{R}^n \to S^1$.

(2a) Natural connections on LM

Pick a Sobolev parameter $s > 1/2$. We put an s-inner product on $T_\gamma LM$ by

$$\langle X, Y\rangle_s = \frac{1}{2\pi}\int_{S^1}\langle (1+\Delta)^s X(\alpha), Y(\alpha)\rangle_{\gamma(\alpha)} d\alpha, \quad X, Y \in \Gamma(\gamma^* TM).$$

Here $\Delta = D^* D$, $D = \frac{D}{d\dot\gamma}$, the covariant derivative along γ.

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Think of s as an annoying regularization parameter. We should study how our theory depends on s, and take the part of the theory that is independent of s.

Natural connections on LM

The Sobolev-s metric makes LM a Riemannian manifold. The Levi-Civita connection ∇^s on LM is determined by

$$\begin{aligned}\langle \nabla^s_Y X, Z\rangle_s &= X\langle Y, Z\rangle_s + Y\langle X, Z\rangle_s - Z\langle X, Y\rangle_s \\ &\quad + \langle [X, Y], Z\rangle_s + \langle [Z, X], Y\rangle_s - \langle [Y, Z], X\rangle_s.\end{aligned}$$

since the right hand side is a *continuous* linear functional of $Z \in T_\gamma LM = \Gamma(\gamma^* TM)$ (for the right topology on the space of sections).

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For $X, Y \in T_\gamma LM$, $\nabla^0_X Y(\gamma)(\theta) = \mathrm{ev}^*_\theta \nabla^{LC,M}_X Y(\gamma)$, *and*

$$\begin{aligned}
&\nabla^1_X Y(\gamma)(\theta) \\
&= \nabla^0_X Y(\gamma)(\theta) + \frac{1}{2}(1+\Delta)^{-1}[-\nabla_{\dot\gamma}(R(X,\dot\gamma)Y)(\theta) \\
&\quad -R(X,\dot\gamma)\nabla_{\dot\gamma}Y(\theta) - \nabla_{\dot\gamma}(R(Y,\dot\gamma)X)(\theta) - R(Y,\dot\gamma)\nabla_{\dot\gamma}X(\theta) \\
&\quad +(R(X,\nabla_{\dot\gamma}Y)\dot\gamma)(\theta) + (R(Y,\nabla_{\dot\gamma}X)\dot\gamma)(\theta)] \\
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\end{aligned}$$

The connection 1-form and curvature 2-form $\omega^1_X \in End(T_\gamma LM) = End(\Gamma(\gamma^* TM)), \Omega^1 = d\omega^1 + \omega^1 \wedge \omega^1$ are *zeroth order* ΨDOs acting on $Y \in T_\gamma LM = \Gamma(S^1 \times \mathbb{R}^n \to S^1)$.

ΨDOs

Let $\Omega \subset \mathbb{R}^n$ be a precompact domain.

For

$$\partial^\alpha = (\partial_{x^1})^{\alpha_1} \cdot \ldots \cdot (\partial_{x^n})^{\alpha_n}, \ \xi^\alpha = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n},$$

let $D = \sum_{|\alpha| \le n_0} a_\alpha(x)\partial^\alpha : C_c^\infty(\Omega) \to C_c^\infty(\Omega)$ be a differential operator. By Fourier transform and Fourier inversion,

$$Df(x) = \int_{T^*\Omega} e^{i(x-y)\cdot\xi} \sigma_D(x,\xi) f(y) \ dy \ d\xi$$

where the *symbol* of D is the polynomial $\sigma_D(x,\xi) = \sum_{|\alpha| \le n_0} \frac{1}{i^{|\alpha|}} a_\alpha(x)\xi^\alpha$. $\sigma_D \sim |\xi|^{n_0}$ as $|\xi| \to \infty$. ΨDOs are defined by the same integral, but with symbol $\sigma(x,\xi) \sim \sum_{k \in \mathbb{Z}_{\ge 0}} a_{n_0-k}(x)|\xi|^{n_0-k}$ growing like $|\xi|^{n_0}$, where the order n_0 of D can be any real number.

ΨDOs

This extends to vector valued operators and then to operators on sections of bundles $E \to M$ over closed manifolds. For $x \in M, \xi \in T^*M$, $\sigma(x, \xi) \in Hom(E_x, E_x)$.

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D is *elliptic* if $\sigma_{n_0}(x, \xi)$ is invertible for $\xi \neq 0$. Standard Laplacian operators are elliptic, with top symbol $\sigma_2(\Delta)(x, \xi) = |\xi|^2 \mathrm{Id}$, as are their inverses (Green's operators), with top symbol $\sigma_{-2}(\Delta^{-1})(x, \xi) = |\xi|^{-2} \mathrm{Id}$.

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Just like DO, $\Psi\mathrm{DO}(E)$ forms a graded algebra, and includes all Green's operators, heat operators, and operators given by smooth kernels. Powers of elliptic operators, like $(1 + \Delta)^s$, are again $\Psi\mathrm{DO}$s.

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Even if an operator is nonlocal, like $(1 + \Delta)^{-1}$, its symbol terms are local/computable.

(2b) Characteristic classes on TLM

For characteristic classes on G-bundles, we need Ad-invariant functions on the Lie algebra $\mathfrak{g}$. The LC connection ∇^1 has connection/curvature forms taking values in $\Psi\mathrm{DO}_{\leq 0} = \mathfrak{g}$, so the structure group is $\Psi\mathrm{DO}_0^*$, the group of invertible zeroth order $\Psi\mathrm{DOs}$. Note that $\Psi\mathrm{DO}_0^* \supset \mathcal{G}$, so we are extending the structure group.

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- the *leading order trace*

$$\mathrm{tr}^{lo}(A) = \int_{S^*N} \mathrm{tr}_x(\sigma_0(A)(x,\xi))\, d\xi\, dx$$

Wodzicki-Chern classes

The theory of characteristic classes carries over to TLM. Let Ω be the curvature of a $\Psi\mathrm{DO}_0^*$-connection on TLM.

Definition:

(i) The i^{th} *Wodzicki-Chern character class* of LM is

$$\begin{aligned} ch_i^W(LM) &= [\mathrm{res}^W(\Omega^i)] \\ &= \left[\int_{S^*S^1} \mathrm{tr}_x(\sigma_{-1}(\Omega^i)(x,\xi))\ d\xi\ dx\right] \\ &\quad \in H^{2i}(LM,\mathbb{C}). \end{aligned}$$

The WCC forms are locally computable.

(3) Chern-Simons classes on TLM

Problem: $ch_i^W(LM) = 0$. Since $ch_i^W(LM)$ is independent of connection, we can compute it for the L^2 connection:

$$ch_i^W(LM) = \left[\int_{S^*S^1} \mathrm{tr}_x(\sigma_{-1}((\Omega^{s=0})^i)(x,\xi))\ d\xi\ dx\right] = [0] = 0.$$

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- *Solution:* Since the ch^W form vanishes pointwise, we can hope to construct Wodzicki-Chern-Simons forms.

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$$ch_i^W(LM) = \left[\int_{S^*S^1} \mathrm{tr}_x(\sigma_{-1}((\Omega^{s=0})^i)(x,\xi))\ d\xi\ dx\right] = [0] = 0.$$

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If Chern character forms vanish for two connections ∇_0, ∇_1 on $E \to N$, then Chern-Simons classes are defined: there is an explicit transgression form $Tch_i \in \Lambda^{2i-1}(N)$ with

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If e.g. ∇_0, ∇_1 are flat, or if $\dim(N) = 2i-1$, then $Tch_i(\nabla_0, \nabla_1)$ is closed and defines the Chern-Simons class

$$CS_i(\nabla_0, \nabla_1) \in H^{2i-1}(N, \mathbb{C}).$$

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LM is infinite dimensional, but the local nature of res^W implies $ch_i^W(\Omega^s) \equiv 0$ as a form if dim $M = 2i - 1$.

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Let $\dim M = 2i-1$. The (2i-1)-Wodzicki-Chern-Simons class is

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Proposition

At a loop $\gamma \in LM$,

$$\begin{aligned} & CS_{2i-1}^W(X_1, ..., X_{2i-1})(\gamma) \\ = \quad & \frac{i}{2^{i-2}} \sum_\sigma \mathrm{sgn}(\sigma) \int_\gamma \mathrm{tr}[(R(X_{\sigma(1)}, \cdot)\dot\gamma)(\Omega^M)^{i-1}(X_{\sigma(2)}, ..., X_{\sigma(2i-1)})]. \end{aligned}$$

Summary of WCS classes

Given (M^{2i-1}, g), we get locally computable classes

$$CS^W_{2i-1}(g) \in H^{2i-1}(LM, \mathbb{C})$$

associated to the $2i$-component of the Chern character.

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Remarks:

We can repeat this construction for any characteristic class of degree $2i$, e.g. some product of Chern classes.

By curvature tensor symmetries, $CS^W_{4i-1}(g) = 0$, so from now on, dim $M = 4i + 1$.

If we use ∇^0, ∇^s instead of ∇^0, ∇^1 in the definition of the WCS form, we just change CS^W to $s \cdot CS^W$.

(3) Relating WCS classes on TLM to Diff(M)

We have a family of Sasakian manifold $\overline{M}_k$ associated to an integral Kähler manifold M^{4i}. $\overline{M}_k$ comes with a natural circle action $a : S^1 \times \overline{M}_k \to \overline{M}_k$.

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Since

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We also get $a^L : \overline{M}_k \to \mathrm{Maps}(S^1, \overline{M}_k) = L\overline{M}_k$ and a class

$$[a^L] = a^L_*[\overline{M}_k] \in H_{4i+1}(L\overline{M}_k, \mathbb{C}).$$

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Fact:

$$[a^L] \neq 0 \Rightarrow [a^D] \neq 0.$$

Using WCS classes to detect elements of $\pi_1(\mathrm{Diff}(\overline{M}_k))$

We have $[a^D] \in \pi_1(\mathrm{Diff}(\overline{M}_k)), [a^L] \in H_{4i+1}(L\overline{M}_k, \mathbb{C})$ with

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- $$\int_{[a^L]} CS^W_{4k+1} \neq 0 \Rightarrow [a^D] \text{ has infinite order.}$$

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- $$\int_{[a^L]} CS^W_{4k+1} \neq 0 \Rightarrow [a^D] \text{ has infinite order.}$$
- $$\int_{[a^L]} CS^W_{4k+1} = \int_{a^L_*[\overline{M}_k]} CS^W_{4k+1} = \int_{\overline{M}_k} a^{L,*} CS^W_{4k+1}$$

is locally computable.

Using WCS classes to detect elements of $\pi_1(\mathrm{Diff}(\overline{M}_k))$

Lemma

Let M be a Kähler surface with local o.n. frame $\{e_2, Je_2, e_3, Je_3\}$ and let $\bar{\xi}$ be the unit vector along the circle fiber of $\overline{M}_k$. Then

$$
\begin{aligned}
& a^{L,*} CS^W_{5,\gamma}(\bar{\xi}, e_2, Je_2, e_3, Je_3) \\
= \quad & \frac{3k^2}{5} \{32\pi^2 p_1(\Omega)(e_2, Je_2, e_3, Je_3) + 32k^2[3R(e_2, Je_2, e_3, Je_3) \\
& -R(e_2, e_3, e_2, e_3) - R(e_2, Je_3, e_2, Je_3) \\
& +R(e_2, Je_2, e_2, Je_2) + R(e_3, Je_3, e_3, Je_3)] \\
& +192k^4\},
\end{aligned}
$$

where $p_1(\Omega)$ is the first Pontrjagin form of M.

Clearly

$$\int_{\overline{M}_k} a^{L,*} CS^W_5 \neq 0 \quad \text{for } k \gg 0.$$

Kähler surfaces

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Theorem

Let (M, ω) be a compact Kähler surface. Then the circle action is an element of infinite order in $\pi_1(\mathrm{Diff}(\overline{M}_k))$ and in $\pi_1(\mathrm{Isom}(\overline{M}_k))$ for $k \gg 0$.

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We can do more careful calculations for specific Kähler surfaces.

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We can do more careful calculations for specific Kähler surfaces.

Proposition

(i) $\pi_1(\mathrm{Diff}(\overline{\mathbb{CP}}^2_k))$ is infinite for $k \neq \pm 1$.
(ii) Let M be a compact projective K3 surface. Then $\pi_1(\mathrm{Diff}(\overline{M}_k))$ is infinite for all k.

S^5 is difficult

Example: There is a family of Sasaki-Einstein metrics $g_a, a \in (0,1)$, on B^5 which match up nicely on ∂B^5 to give metrics on $S^2 \times S^3$. We get

$$\int_{[a^L]} CS_5^W(g_a) = -\frac{1849\pi^4}{37750}(-1 + a^2),$$

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At $a = 1$, the metric glues up to the standard metric on S^5. But now we conclude nothing about $\pi_1(\mathrm{Diff}(S^5))$.

Higher dimensions

For any r, on $\overline{M}_k$

$$a^* CS^W_{4r+1}(\gamma) = \sum_{i=1}^{2r} \alpha_i k^{2i} = \alpha_1 k^2 + \sum_{i=2}^{2r} \alpha_i k^{2i}$$

with $\alpha_i \in \Lambda^{4r+1}(\overline{M}_k)$.

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Lemma

$\alpha_1(\bar{\xi}, \cdot)$ is a multiple of $ch_{2r}(\Omega^M)$.

As before, if $[ch_{2r}] \neq 0 \in H^{2r}(M)$, then for some cycle $[\sigma] \in H_{2r}(M)$,

$$\int_\sigma a^* CS^W_{4r+1}(\gamma) = \sum \left(\int_\sigma \alpha_i \right) k^{2i} = \left(\int_\sigma \alpha_1 \right) k^2 + h.o. \neq 0$$

for $k \gg 0$. As before, this implies $\pi_1(\mathrm{Diff}(\overline{M}_k))$ is infinite.

Higher dimensions

Theorem

(i) $\pi_1(\mathrm{Diff}(\overline{\mathbb{CP}}_k^{2i}))$ *is infinite for* $k \gg 0$.
(ii) Let M *have real dimension* $4i$*. If* $\sigma(M) \neq 0$*, then* $\pi_1(\mathrm{Diff}(\overline{M}_k))$ *is infinite for* $k \gg 0$.

Proof: If $\sigma(M) \neq 0$, then some Pontrjagin number is nonzero, which implies that some Chern character component ch_{2r} is nonzero.

Future directions

- Every symplectic manifold is "Kähler except for integrability." Do these results carry over for line bundles over integral symplectic manifolds?
- Find a nonstandard metric on S^5 such that $\int_{[a^L]} CS_5^W(g) \neq 0$, or prove that no such metric exists.

References for most of this material

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