T-duality and Atiyah duality

Craig Westerland

University of Melbourne

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Twisted K-theory

Twisted K-theory is an invariant associated to a topological space X and a class $H \in H^3(X) = [X, K(\mathbb{Z}, 3)]$.

Definition

For a Hilbert space $\mathcal{H}$, $\mathrm{PU}(\mathcal{H})$ is the projective unitary group of $\mathcal{H}$ and $\mathrm{Fred}(\mathcal{H})$ is the space of Fredholm operators on $\mathcal{H}$.

Kuiper's theorem implies that $\mathrm{PU}(\mathcal{H}) = K(\mathbb{Z}, 2)$, and thus the classifying space $B\,\mathrm{PU}(\mathcal{H}) = K(\mathbb{Z}, 3)$.

Recall that $\mathrm{Fred}(\mathcal{H}) \simeq BU \times \mathbb{Z}$ is a representing space for K-theory:

$$K^0(X) = [X, \mathrm{Fred}(\mathcal{H})].$$

Note further that $\mathrm{PU}(\mathcal{H})$ acts naturally on $\mathrm{Fred}(\mathcal{H})$.

Construction

Define Φ_H – a principal $\mathrm{PU}(\mathcal{H})$-bundle over X – via pullback over H of the tautological bundle $E\,\mathrm{PU}(\mathcal{H}) \to B\,\mathrm{PU}(\mathcal{H})$:

$$\begin{tikzcd} \Phi_H \arrow[r] \arrow[d] & E\,\mathrm{PU}(\mathcal{H}) \arrow[d] \\ X \arrow[r, "H"] & B\,\mathrm{PU}(\mathcal{H}) = K(\mathbb{Z},3) \end{tikzcd}$$

There is an associated bundle $\Phi_H \times_{\mathrm{PU}(\mathcal{H})} \mathrm{Fred}(\mathcal{H})$ over X with fibre $\mathrm{Fred}(\mathcal{H})$.

Definition

The *twisted K-theory* of X

$$K^0_H(X) := \Gamma[X; \Phi_H \times_{\mathrm{PU}(\mathcal{H})} \mathrm{Fred}(\mathcal{H})]$$

is the group of homotopy classes of sections of this bundle.

Twisted K-theory spectra

Recall that *spectra* are objects in the *stable homotopy category*. This enlarges the category of spaces by allowing *desuspension* of spaces.

Objects in this category define cohomology theories as representable functors (e.g., $K^*(X) = [X_+, \mathcal{K}]$ where $\mathcal{K}$ is the spectrum representing K-theory).

For a space X and $H \in H^3(X)$, define spectra $\mathcal{K}_H(X)$ and $\mathcal{K}^H(X)$ so

$$\pi_{-*}\mathcal{K}_H(X) = K_H^*(X) \quad \text{and} \quad \pi_*\mathcal{K}^H(X) = K_*^H(X).$$

If $H = 0$, then $\mathcal{K}_H(X) = F(X_+, \mathcal{K})$ and $\mathcal{K}^H(X) = X_+ \wedge \mathcal{K}$. The description above makes $\mathcal{K}$ into a $\mathrm{PU}(\mathcal{H})$-equivariant spectrum.

The homomorphism $\Omega H : \Omega X \to K(\mathbb{Z}, 2) = \mathrm{PU}(\mathcal{H})$, gives ΩX an action on $\mathcal{K}$. The *homotopy fixed point* and *orbit spectra* are twisted K-theory:

Proposition

$$\mathcal{K}_H(X) \simeq \mathcal{K}^{h\Omega X} = F^{\Omega X}(E\Omega X_+, \mathcal{K}) \quad \text{and} \quad \mathcal{K}^H(X) \simeq \mathcal{K}_{h\Omega X} = \mathcal{K} \wedge_{\Omega X} E\Omega X_+.$$

T-dual circle bundles

Let $\pi : E \to M$ and $\widehat{\pi} : \widehat{E} \to M$ be principal circle bundles over a manifold M. Form the *correspondence space*:

$$\begin{tikzcd}
 & E \times_M \widehat{E} \arrow[dl, "p"'] \arrow[dr, "\widehat{p}"] & \\
E \arrow[dr, "\pi"'] & & \widehat{E} \arrow[dl, "\widehat{\pi}"] \\
 & M &
\end{tikzcd}$$

Definition

For $H \in H^3(E)$ and $\widehat{H} \in H^3(\widehat{E})$, the *pairs* (E, H) and $(\widehat{E}, \widehat{H})$ are *T-dual* if

$$\pi_!(H) = c_1(\widehat{E}), \quad \widehat{\pi}_!(\widehat{H}) = c_1(E), \quad \text{and} \quad p^*(H) = \widehat{p}^*(\widehat{H}).$$

The T-duality isomorphism

Bouwknegt-Evslin-Mathai show:

Theorem (BEM '04)

If (E, H) and $(\widehat{E}, \widehat{H})$ are T-dual, there is a map

$$\lambda^* : K^*_{p^*(H)}(E \times_M \widehat{E}) \to K^*_{\widehat{p}^*(\widehat{H})}(E \times_M \widehat{E})$$

with the property that the composite

$$\widehat{p}_! \circ \lambda^* \circ p^* : K^*_H(E) \to K^{*+1}_{\widehat{H}}(\widehat{E})$$

is an isomorphism.

Goal: Understand and explain this result using stable homotopy theory.

Circle bundles over S^2

Let $M = S^2$, and consider the Hopf fibration $\eta : S^3 \to S^2$ and its reduction $\overline{\eta} : \mathbb{R}P^3 \to S^2$. These bundles are classified by the Chern classes $1, 2 \in \mathbb{Z} = H^2(S^2)$.

Note that $H^3(S^3) = \mathbb{Z} = H^3(\mathbb{R}P^3)$, so

$$(S^3, H = 2) \text{ and } (\mathbb{R}P^3, \widehat{H} = 1) \text{ are T-dual.}$$

We verify [BEM] by computing:

$$\begin{array}{lcl} K^0_H(S^3) = 0, & & K^1_H(S^3) = \mathbb{Z}/2, \\ & \text{while} & \\ K^0_{\widehat{H}}(\mathbb{R}P^3) = \mathbb{Z}/2, & & K^1_{\widehat{H}}(\mathbb{R}P^3) = 0 \end{array}$$

using the Atiyah-Hirzebruch spectral sequence for twisted K-theory.

A warm-up

Let M be a point; the correspondence diagram becomes

$$S^1 \xleftarrow{p} S^1 \times \widehat{S}^1 \xrightarrow{\widehat{p}} \widehat{S}^1$$

We compute:

$$K^*(S^1) = \Lambda[x], \quad K^*(\widehat{S}^1) = \Lambda[\widehat{x}], \quad \text{and} \quad K^*(S^1 \times \widehat{S}^1) = \Lambda[x, \widehat{x}].$$

Further, $\lambda^*(z) = z \otimes P$, where $P \to S^1 \times \widehat{S}^1$ is the *Poincaré line bundle*. P is isomorphic to the pullback of the tautological bundle over $S^2 = \mathbb{C}P^1$ via the map that collapses the 1-skeleton.

In $K^*(S^1 \times \widehat{S}^1)$, $P = 1 + x\widehat{x}$. We verify that $\widehat{p}_! \circ \lambda^* \circ p^*$ is an isomorphism:

$$\begin{array}{c} \Lambda[x] \xrightarrow{p^*} \Lambda[x, \widehat{x}] \xrightarrow{\lambda^*} \Lambda[x, \widehat{x}] \xrightarrow{\widehat{p}_!} \Lambda[\widehat{x}]. \\ 1 \longrightarrow 1 \longrightarrow 1 + x\widehat{x} \longrightarrow \widehat{x} \\ x \longrightarrow x \longrightarrow x(1 + x\widehat{x}) = x \longrightarrow 1 \end{array}$$

The idea

For the bundle $\pi : E \to M$, we have a fibre sequence:

$$\cdots \longrightarrow \Omega E \xrightarrow{\Omega\pi} \Omega M \longrightarrow S^1 \longrightarrow E \xrightarrow{\pi} M$$

which makes S^1 a space with an ΩM action. Further, the homotopy quotient

$$S^1_{h\Omega M} = [S^1/\Omega M] = E$$

There are similar descriptions of $\widehat{E}$ and $E \times_M \widehat{E}$ as homotopy quotients of $\widehat{S}^1$ and $S^1 \times \widehat{S}^1$ by ΩM.

Plan: We will reprove the T-duality isomorphism by extending the previous isomorphism ΩM-equivariantly. The subtlety is incorporating the twisting.

The space R

Definition

Define R to be the homotopy fibre of the map

$$c \cup \widehat{c} : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 4)$$

defining the product $c \cup \widehat{c} \in H^4(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2))$ of first Chern classes in $H^2(K(\mathbb{Z}, 2))$.

Theorem (Bunke-Schick, '05)

R is a classifying space for pairs (E, H):

$$[M, R] = \{(E, H) \mid \pi : E \to M \text{ an } S^1\text{-bundle, } H \in H^3(E)\}/\cong$$

Why: The Gysin sequence implies that $c_1(E) \cup \pi_!(H) = 0$.

Universal bundles

Extend the defining fibre sequence:

$$\mathrm{PU}(\mathcal{H}) \longrightarrow \Omega R \longrightarrow S^1 \times \widehat{S}^1 \xrightarrow{0} K(\mathbb{Z},3) \longrightarrow R \longrightarrow K(\mathbb{Z},2) \times K(\mathbb{Z},2) \xrightarrow{c \cup \widehat{c}} K(\mathbb{Z},4).$$

Then if (E, H) corresponds to $f : M \to R$, we may pull this back over f:

$$\begin{tikzcd}
\Omega R \arrow[r] & S^1 \times \widehat{S}^1 \arrow[r] & K(\mathbb{Z},3) \arrow[r] & R \\
\Omega M \arrow[r] \arrow[u, "\Omega f"] & S^1 \times \widehat{S}^1 \arrow[r] \arrow[u, "="'] & E \times_M \widehat{E} \arrow[r] \arrow[u] & M, \arrow[u, "f"']
\end{tikzcd}$$

Define $S := K(\mathbb{Z},3)/\widehat{S}^1$ and $\widehat{S} := K(\mathbb{Z},3)/S^1$.

These are circle bundles over R; pulling them back over f gives E and $\widehat{E}$, respectively. Further, Ωf lifts:

$$\begin{tikzcd}
 & \Omega S \arrow[d] \\
\Omega M \arrow[ur] \arrow[r, "\Omega f"] & \Omega R
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Omega \widehat{S} \arrow[d] & \\
\Omega R & \Omega M \arrow[l, "\Omega f"'] \arrow[ul]
\end{tikzcd}$$

Summary

1. $S^1 \times \widehat{S}^1 = \Omega R / \mathrm{PU}(\mathcal{H})$. So $S^1 \times \widehat{S}^1$ is an ΩR-space.
2. $\Omega f : \Omega M \to \Omega R$ is a homomorphism, as are the lifts to ΩS and $\Omega \widehat{S}$.
3. $E \times_M \widehat{E}$ is the homotopy orbit space

$$E \times_M \widehat{E} = (S^1 \times \widehat{S}^1)_{h\Omega M} = \Omega M \backslash \Omega R / \mathrm{PU}(\mathcal{H}).$$

4. Similarly,

$$E = S^1_{h\Omega M} = \Omega M \backslash \Omega S / \mathrm{PU}(\mathcal{H}) \quad \text{and} \quad \widehat{E} = \widehat{S}^1_{h\Omega M} = \Omega M \backslash \Omega \widehat{S} / \mathrm{PU}(\mathcal{H})$$

Corollary

There are equivalences

$$\mathcal{K}^{H}(E) \simeq (\Omega S_+ \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K})_{h\Omega M}, \quad \mathcal{K}^{\widehat{H}}(\widehat{E}) \simeq (\Omega \widehat{S}_+ \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K})_{h\Omega M},$$

and $\mathcal{K}^{p^*(H)}(E \times_M \widehat{E}) \simeq (\Omega R_+ \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K})_{h\Omega M}$.

Definition

Let

$$\overline{\lambda} : \Omega R_+ \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K} \to \Omega R_+ \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K}$$

be right multiplication in $\mathcal{K}$ by $1 + \beta$, where β is the Bott class.

Since $\Omega R / \mathrm{PU}(\mathcal{H}) = S^1 \times \widehat{S}^1$, $\overline{\lambda}$ induces multiplication by P in $K^*(S^1 \times \widehat{S}^1)$. $\overline{\lambda}$ is equivariant for the action of ΩM, so descends to

$$\lambda : \mathcal{K}^{p^*(H)}(E \times_M \widehat{E}) \to \mathcal{K}^{p^*(H)}(E \times_M \widehat{E}) \simeq \mathcal{K}^{\widehat{p}^*(\widehat{H})}(E \times_M \widehat{E})$$

We therefore get the T-duality isomorphism ΩM-equivariantly from

$$\begin{tikzcd}
\Sigma(\Omega S_+ \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K}) \arrow[r, "p^!"] & \Omega R_+ \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K} \arrow[r, "\overline{\lambda}"] & \Omega R \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K} \arrow[r, "\widehat{p}"] & \Omega \widehat{S}_+ \wedge_{\mathrm{PU}(\mathcal{H})} \mathcal{K} \\
\Sigma(S^1_+ \wedge \mathcal{K}) \arrow[r, "p^!"] \arrow[u, "\simeq"'] & (S^1 \times \widehat{S}^1)_+ \wedge \mathcal{K} \arrow[r, "1 \wedge P"] \arrow[u, "\simeq"'] & (S^1 \times \widehat{S}^1)_+ \wedge \mathcal{K} \arrow[r, "\widehat{p}"] \arrow[u, "\simeq"'] & \widehat{S}^1_+ \wedge \mathcal{K} \arrow[u, "\simeq"']
\end{tikzcd}$$

which we saw was an equivalence in the warm-up.

Poincaré duality

Let h_* be a homology theory and let M be a closed manifold which is orientable with respect to h_*. Examine the diagram

$$M \times M \xleftarrow{\Delta} M \xrightarrow{c} pt,$$

Orientability of M provides a (Pontrjagin-Thom) shriek map for Δ:

$$h_*(M) \otimes_{h_*} h_*(M) \xrightarrow{\times} h_*(M \times M) \xrightarrow{\Delta^!} h_{*-\dim M}(M) \xrightarrow{c_*} h_{*-\dim M}(pt).$$

This is adjoint to a map $h_*(M) \to \mathrm{Hom}_{h_*}(h_{\dim M - *}(M), h_*)$. When $h_* = H_*(\cdot; \mathbb{F})$ is singular homology with field coefficients, the target is $\mathrm{Hom}_{\mathbb{F}}(H_{\dim M - *}(M), \mathbb{F}) \cong H^{\dim M - *}(M)$; this is the Poincaré duality isomorphism.

When h_* is a homology theory consisting of geometric cycles, the composite of the first two maps is interpreted as the transversal intersection of those cycles in M, and is written

$$x \pitchfork y := \Delta^!(x \times y).$$

Atiyah duality

If $e : M \to \mathbb{R}^N$ is an embedding with normal bundle ν, we will write M^{-TM} for the Thom spectrum

$$M^{-TM} := \Sigma^{-N} M^{\nu},$$

the desuspension of the Thom space of ν.

The Pontrjagin-Thom collapse map for Δ is $\Delta^! : M \times M \to M^{TM}$, where TM is the tangent bundle of M (masquerading as the normal bundle of Δ), and M^{TM} is its Thom space. Adding ν and desuspending by N, we get an intersection pairing

$$M_+ \wedge M^{-TM} \xrightarrow{\Delta^!} M_+ \xrightarrow{c} S.$$

The adjoint map

$$a : M^{-TM} \to F(M_+, S)$$

is an equivalence; this is *Atiyah duality*. It implies h-Poincaré duality when M is h-orientable.

An intersection pairing for T-dual bundles

Consider the diagram

$$\begin{tikzcd} E \times \widehat{E} \arrow[d, "\pi \times \widehat{\pi}"'] & E \times_M \widehat{E} \arrow[l, "\widehat{\Delta}"'] \arrow[d, "\tilde{\pi}"'] \arrow[dr, "c"] & \\ M \times M & M \arrow[l, "\Delta"'] \arrow[r, "c"] & pt. \end{tikzcd} \tag{1}$$

There is an umkehr map

$$\widehat{\Delta}^! : \mathcal{K}^{(H,-\widehat{H})}(E \times \widehat{E}) \to \mathcal{K}^{\widehat{\Delta}^*(H,-\widehat{H})}(E \times_M \widehat{E})^{TM}.$$

Notice that $\widehat{\Delta} = (p, \widehat{p})$. Thus $\widehat{\Delta}^*(H, -\widehat{H}) = p^*(H) - \widehat{p}^*(\widehat{H}) = 0$. So $\mathcal{K}^{\widehat{\Delta}^*(H,-\widehat{H})}(E \times_M \widehat{E}) \simeq \mathcal{K} \wedge (E \times_M \widehat{E}_+)$. Further,

$$\mathcal{K}^{(H,-\widehat{H})}(E \times \widehat{E}) \simeq \mathcal{K}^H(E) \wedge_{\mathcal{K}} \mathcal{K}^{-\widehat{H}}(\widehat{E}).$$

Desuspending by TM, we may form the composite

$$\mu : \mathcal{K}^H(E)^{-TM} \wedge_{\mathcal{K}} \mathcal{K}^{-\widehat{H}}(\widehat{E}) \xrightarrow{\widehat{\Delta}^!} \mathcal{K} \wedge (E \times_M \widehat{E}_+) \xrightarrow{(1+\beta)\wedge 1} \mathcal{K} \wedge (E \times_M \widehat{E}_+) \xrightarrow{1 \wedge c} \mathcal{K}.$$

Theorem

The adjoint map

$$t : \mathcal{K}^H(E)^{-TM} \to F_{\mathcal{K}}(\mathcal{K}^{-\widehat{H}}(\widehat{E}), \mathcal{K})$$

to μ is an equivalence.

This is precisely the T-duality isomorphism when M is spin$_{\mathbb{C}}$.

Remark

This allows us to think of μ as a nondegenerate pairing of $\mathcal{K}$-modules. However, μ_* is *not* in general nondegenerate; e.g.:

$$\mu_* : K_*^H(S^3) \otimes_{K_*} K_*^{-\widehat{H}}(\mathbb{R}P^3) \to K_*$$

is degenerate, since the domain is $\mathbb{Z}/2 \otimes \mathbb{Z}/2$, and the target is $\mathbb{Z}$ (in different degrees, no less!).

A similar phenomenon occurs for torsion in singular cohomology, but it is impossible for the *entire* cohomology ring to be torsion.

Questions

1. Can we prove similar results after changing the fibres from S^1 to other compact Lie groups?
2. Does this work at other cohomology theories, especially localised K-theory, Morava K-theory, TMF?
3. In p-local K-theory, S^{2p-3} becomes a (p-compact) group [Sullivan]. Does the same argument go through?
4. At Morava $K(n)$-theory, $K(\pi, q)$ is a ($K(n)$-compact) group for $q < n$ and π a finite group [Ravenel–Wilson]...
5. What do these “exotic” intersection pairings tell us about the K-local category?

Thanks

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