

Dirac Operators in Geometry, Topology, Representation Theory, and Physics

Kan extension
and
classification
theorems

Michael
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Topological
field theories,
classical
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Internal Kan
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Applications

*All concepts are Kan extensions.
S. MacLane.*

Kan extension and classification theorems

Michael Batatin

21 October 2010

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The category Cob_n of n -cobordisms:

- objects: oriented $(n - 1)$ -dimensional manifolds;
- morphisms: oriented cobordisms (up to diffeomorphism) between them.

The category Cob_n is symmetric monoidal.

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The category Cob_n of n -cobordisms:

- objects: oriented $(n - 1)$ -dimensional manifolds;
- morphisms: oriented cobordisms (up to diffeomorphism) between them.

The category Cob_n is symmetric monoidal.

Definition

An n -dimensional TFT is a symmetric monoidal functor

$$Z : Cob_n \rightarrow Vect$$

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Classification of TFT in dim 1 and 2:

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Classification of TFT in dim 1 and 2:

- a 1-dimensional TFT can be reconstructed from a finite dimensional vector space;

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Classification of TFT in dim 1 and 2:

- a 1-dimensional TFT can be reconstructed from a finite dimensional vector space;
- a 2-dimensional TFT can be reconstructed from a commutative Frobenius algebra.

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Classification of TFT in dim 1 and 2:

- a 1-dimensional TFT can be reconstructed from a finite dimensional vector space;
- a 2-dimensional TFT can be reconstructed from a commutative Frobenius algebra.
- (an open version of TFT) an open 2-dimensional TFT can be reconstructed from a noncommutative Frobenius algebra.

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Proof.

- $n = 1$. Restrict Z to the subcategory of connected 1-manifolds and prove that the rest of Z can be reconstructed from this restriction.

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Proof.

- $n = 1$. Restrict Z to the subcategory of connected 1-manifolds and prove that the rest of Z can be reconstructed from this restriction.
- $n = 2$. Restrict Z to the subcategory of 0-genus surfaces and proved that the rest of Z can be reconstructed from this restriction.

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Proof.

- $n = 1$. Restrict Z to the subcategory of connected 1-manifolds and prove that the rest of Z can be reconstructed from this restriction.
- $n = 2$. Restrict Z to the subcategory of 0-genus surfaces and proved that the rest of Z can be reconstructed from this restriction.

The subcategory of 0-genus surfaces is PROP for commutative Frobenius algebras.

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The situation reminds the situation in representation theory for induced and restricted representation.

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The situation reminds the situation in representation theory for induced and restricted representation.

Given a subgroup $i : H \subset G$ we have a restriction functor

$$res : Rep(G) \rightarrow Rep(H)$$

which has a left adjoint:

$$Ind : Rep(H) \rightarrow Rep(G).$$

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More generally, given a morphism between algebras

$$f : A \rightarrow B$$

we have two adjoint functors

$$f^* : \text{Mod}(B) \rightarrow \text{Mod}(A)$$

$$f_! : \text{Mod}(A) \rightarrow \text{Mod}(B).$$

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More generally, given a morphism between algebras

$$f : A \rightarrow B$$

we have two adjoint functors

$$f^* : \text{Mod}(B) \rightarrow \text{Mod}(A)$$

$$f_! : \text{Mod}(A) \rightarrow \text{Mod}(B).$$

Many classification results amount to characterisation of the image of $f_!$ (descent theory).

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Let $f : C \rightarrow D$ be a functor between two (small) categories.
Let V be another category. There is an induced functor

$$\text{res}_f : [D, V] \rightarrow [C, V].$$

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Let $f : C \rightarrow D$ be a functor between two (small) categories.
Let V be another category. There is an induced functor

$$res_f : [D, V] \rightarrow [C, V].$$

Definition

A left Kan extension

$$Lan_f : [C, V] \rightarrow [D, V]$$

is a left adjoint functor (if exists) to res_f .

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Definition

A left Kan extension

$$\text{Lan}_f : [C, V] \rightarrow [D, V]$$

is a left adjoint functor (if exists) to res_f .

Example 1. If $D = \Sigma G, C = \Sigma H, V = \text{Vect}$ then $\text{Ind} = \text{Lan}_{\Sigma i}$.

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Example 2. If $f : A \rightarrow B$ is a homomorphism of algebras considered as a one object k -linear category.

$$\text{Lan}_f(M) = f_!(M) = B \otimes_A M.$$

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Example 2. If $f : A \rightarrow B$ is a homomorphism of algebras considered as a one object k -linear category.

$$\mathit{Lan}_f(M) = f_!(M) = B \otimes_A M.$$

Classical pointwise formula for left Kan extension:

$$\mathit{Lan}_f(g)(d) = \mathit{colim}_{f/d} g^*(-).$$

Here, f/d is the slice category of f over d (homotopy or lax-fiber of f over d .)

$g^* : f/d \rightarrow V$ is the restriction of g on this fiber.

Kan extensions in 2-categories.

Classical Kan Extension is not enough for classification results in TFT because the kind of functors involved are symmetric monoidal.

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Generalisation. Recall that a 2-category is the same as *Cat*-enriched category.

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Generalisation. Recall that a 2-category is the same as *Cat*-enriched category.

Let K be a 2-category. And let $f : C \rightarrow D$ be a 1-cell in K . For any $V \in K$ it determines a functor

$$res : K(D, V) \rightarrow K(C, V).$$

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Let K be a 2-category. And let $f : C \rightarrow D$ be a 1-cell in K . For any $V \in K$ it determines a functor

$$res : K(D, V) \rightarrow K(C, V).$$

Definition

A left adjoint to *res* is called left Kan extension along f . For a given $g : C \rightarrow V$ the left Kan extension satisfies

$$K(Lan_f(g), h) \simeq K(g, f \cdot h)$$



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- If $K = \mathit{Cat}$ the left Kan extension is classical left Kan extension.

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- If $K = \mathit{Cat}$ the left Kan extension is classical left Kan extension.
- If $K = \mathit{SymCat}$ the category of symmetric monoidal categories, symmetric monoidal functors and symmetric monoidal transformation the Lan_f is precisely the kind of extension which we need for TFT classification results.

Kan extensions in 2-categories.

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Applications

- If $K = \mathit{Cat}$ the left Kan extension is classical left Kan extension.
 - If $K = \mathit{SymCat}$ the category of symmetric monoidal categories, symmetric monoidal functors and symmetric monoidal transformation the Lan_f is precisely the kind of extension which we need for TFT classification results.
- It may be a difficult task to compute Kan extensions in 2-categories more general than Cat .

Kan extensions in 2-categories.

We will mostly interested in 2-categories which are coming as models of some categorified algebraic theories.

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We will mostly interested in 2-categories which are coming as models of some categorified algebraic theories.

Motivating example. The symmetric monoidal categories of cobordisms in the definition of two dimensional TFT is a PROP.

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We will mostly interested in 2-categories which are coming as models of some categorified algebraic theories.

Motivating example. The symmetric monoidal categories of cobordisms in the definition of two dimensional TFT is a PROP.

- **Miracle!**: in categorified algebraic theories we can compute Kan extensions using the same classical formulas as in *Cat*. This is particular true for operadic algebraic theories.

We will see that there is a general method for explicitly computing left adjoint functors using Kan extensions in categorified algebraic theories.

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Applications

Let me start from a very general type of theories:
let C be a category and $T : C \rightarrow C$ be a functor.

Definition

A monad structure on T is a pair of natural transformations:
 $\epsilon : Id \rightarrow T$ and $\mu : T^2 \rightarrow T$ which satisfy
 $\mu \cdot (\mu \otimes 1) = \mu \cdot (1 \otimes \mu)$ and $\mu \cdot (\epsilon \otimes 1) = \mu \cdot (1 \otimes \epsilon)$.

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 $\mu \cdot (\mu \otimes 1) = \mu \cdot (1 \otimes \mu)$ and $\mu \cdot (\epsilon \otimes 1) = \mu \cdot (1 \otimes \epsilon)$.

Definition

An algebra of T is an object $X \in C$ together with a morphism
 $k : TX \rightarrow X$ which satisfy :

$$\epsilon \cdot k = Id_X \quad , \quad k \cdot \mu = k \cdot k.$$

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A monad T is called *cartesian* if C has finite limits, T preserves pullbacks and all naturality squares for unit and multiplication of T are pullbacks.

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A monad T is called *cartesian* if C has finite limits, T preserves pullbacks and all naturality squares for unit and multiplication of T are pullbacks.

Observation. If T is cartesian then there is a canonical "categorification" T_c of T : this is a monad in the 2-category of internal categories $Cat(C)$. It extends T from C considered as a subcategory of discrete objects of $Cat(C)$.

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Observation. If T is cartesian then there is a canonical "categorification" T_c of T : this is a monad in the 2-category of internal categories $Cat(C)$. It extends T from C considered as a subcategory of discrete objects of $Cat(C)$.

Example. Let $C = Set$ and let T be free monoid monad. T is cartesian and $Cat(Set) = Set$. The monad T_c has strict monoidal categories as its category of algebras.

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An algebra of T_c is called **categorical algebra of T** .

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Applications

An algebra of T_c is called **categorical algebra of T** .

Definition

An internal algebra of a monad T inside a categorical algebra M of T is a lax-morphism of categorical T -algebras

$$1 \rightarrow M.$$

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An algebra of T_c is called **categorical algebra of T** .

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Example 1. For the free monoid monad T an internal algebra in a categorical algebra M is the same as a monoid in the monoidal category M .

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An algebra of T_c is called **categorical algebra of T** .

Definition

An internal algebra of a monad T inside a categorical algebra M of T is a lax-morphism of categorical T -algebras

$$1 \rightarrow M.$$

Example 1. For the free monoid monad T an internal algebra in a categorical algebra M is the same as a monoid in the monoidal category M .

Example 2. Let C be a small category. There is a cartesian monad T whose category of algebras are exactly $[C, Set]$. A categorical algebra is a functor $F : C \rightarrow Cat$. An internal algebra is a section of Grothendieck construction $\int F$.

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Fix a categorical algebra M . The internal algebras in M form a category. This gives a 2-functor:

$$\text{Int} : \text{Cat}(\text{Alg}_T) \rightarrow \text{Cat}$$

Theorem

The 2-functor is representable. The representing object H^T is given by a codescent object of the following diagram of categorical T -algebras :

$$T(1) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} T^2(1) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} T^3(1)$$

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Example 1. For the free monoid monad T the categorical T -algebra H^T is the category of all finite ordinals Δ .

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Example 1. For the free monoid monad T the categorical T -algebra H^T is the category of all finite ordinals Δ .

Example 2. For the free nonsymmetric operad monad the algebra H^T is Stasheff's operad of trees. Its geometric realization is operad of associahedra.

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Example 1. For the free monoid monad T the categorical T -algebra H^T is the category of all finite ordinals Δ .

Example 2. For the free nonsymmetric operad monad the algebra H^T is Stasheff's operad of trees. Its geometric realization is operad of associahedra.

Example 3. For the free cyclic operad the algebra H^T is the operad of planar cyclic graphs. Its geometric realization is Stasheff's cyclic A_∞ -operad (topological).

Geometric morphisms and Kan extensions.

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Let T be a cartesian monad on a category C and let S be a cartesian monad on a category D .

Geometric morphisms and Kan extensions.

Let T be a cartesian monad on a category C and let S be a cartesian monad on a category D .

Definition

A geometric morphism between categories of algebras of T and S is a pair (r, R) of right adjoint functors

$$r : C \rightarrow D, R : Alg_T \rightarrow Alg_S$$

such that the following Kan square of right adjoints commutes:

$$\begin{array}{ccc} Alg_T & \xrightarrow{R} & Alg_S \\ \downarrow U_T & & \downarrow U_S \\ C & \xrightarrow{r} & D \end{array}$$

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Let $(r, R) : Alg_T \rightarrow Alg_S$ be a geometric morphism. It induces a right adjoint between categorical algebras:

$$R_c : Cat(Alg_T) \rightarrow Cat(Alg_S).$$

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$$R_c : Cat(Alg_T) \rightarrow Cat(Alg_S).$$

Definition

Let A be a categorical T -algebra. An internal S -algebra in A is an internal S -algebra in the categorical S -algebra $R_c(A)$.

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$$R_c : Cat(Alg_T) \rightarrow Cat(Alg_S).$$

Definition

Let A be a categorical T -algebra. An internal S -algebra in A is an internal S -algebra in the categorical S -algebra $R_c(A)$.

Example. A monoid in a symmetric monoidal category is an internal algebra of free monoid monad inside a categorical algebra of free commutative monoid.

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Let $Int_S : Alg_T \rightarrow Cat$ be the functor of internal S -algebras.

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Let $Int_S : Alg_T \rightarrow Cat$ be the functor of internal S -algebras.

Theorem

The functor Int_S is representable. The representable object h^S is given by a codescent object of the following diagram of categorical T -algebras:

$$G(1) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} GT(1) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} GT^2(1)$$

Here G is the composite

$$D \xrightarrow{I} C \xrightarrow{F_T} Alg_T,$$

I is left adjoint to r and F_T is free T -algebra functor.

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This theorem says that one can replace any internal S -algebra A in a categorical T -algebra V by a T -functor

$$\tilde{A} : h^S \rightarrow V.$$

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This theorem says that one can replace any internal S -algebra A in a categorical T -algebra V by a T -functor

$$\tilde{A} : h^S \rightarrow V.$$

There is also a canonical T -functor

$$f^T : h^S \rightarrow H^T$$

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This theorem says that one can replace any internal S -algebra A in a categorical T -algebra V by a T -functor

$$\tilde{A} : h^S \rightarrow V.$$

There is also a canonical T -functor

$$f^T : h^S \rightarrow H^T$$

Theorem

Let (r, R) be a geometric morphism. Then the left adjoint to R

$$L : \text{Algs} \rightarrow \text{Alg}_T$$

can be computed as a Kan extension of \tilde{A} along f^T in the 2-category of categorical T -algebras.

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Corollary

$$L(A) = \operatorname{colim}_{h^S} \tilde{A}.$$

Proof. The categorical T -algebra H^S has a terminal object and it is generated by it. Therefore, the calculation of Kan extension amounts to the calculation of the colimit over the fiber over 1 which is the whole h^S .

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This formula is especially useful for calculation of left derived functors. In this case one can replace colimit by homotopy colimit.

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If we are interested in calculation of left derived functor on a terminal S -algebra we have

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This formula is especially useful for calculation of left derived functors. In this case one can replace colimit by homotopy colimit.

If we are interested in calculation of left derived functor on a terminal S -algebra we have

Theorem

The left derived functor of L on 1 is given by bar construction

$$B(G, T, 1).$$

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Let O_1 be a category of some sort of operads and O_2 be another category of operads and let $R : O_1 \rightarrow O_2$ be a right adjoint functor.

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Let O_1 be a category of some sort of operads and O_2 be another category of operads and let $R : O_1 \rightarrow O_2$ be a right adjoint functor.

Let also V be another category such that for the objects of V one can consider endomorphism operads $End_{O_1}(X) \in O_1$, and $End_{O_2}(X) \in O_2$.

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Let O_1 be a category of some sort of operads and O_2 be another category of operads and let $R : O_1 \rightarrow O_2$ be a right adjoint functor.

Let also V be another category such that for the objects of V one can consider endomorphism operads $End_{O_1}(X) \in O_1$, and $End_{O_2}(X) \in O_2$.

Moreover, let R preserves the endomorphism operads i.e.

$$End_{O_2}(X) \simeq REnd_{O_1}(X).$$

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Let $A \in \mathcal{O}_2$. Then an A -algebra structure on X is given by

$$k : A \rightarrow \text{End}_{\mathcal{O}_2}(X) \simeq \text{REnd}_{\mathcal{O}_1}(X).$$

By adjunction this gives

$$L(A) \rightarrow \text{End}_{\mathcal{O}_1}(X).$$

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Therefore, the category of A -algebras in V is isomorphic to the category of $L(A)$ -algebras in V . So, if we are able to recognise an operad $B \in O_1$ as $L(A)$ we can classify B -algebras as A -algebras.

Many classification results follows from this simple fact.

Two dimensional TFT

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Alternative definition of 2-dimensional TFT: it is an algebra over modular operad of two dimensional surfaces.

There is a forgetful functor

$$U : ModO(V) \rightarrow CycO(V).$$

This is geometric morphism with left adjoint Mod .

It is not difficult to see that the modular operad $Mod(1)$ is exactly the operad of 2-dimensional surfaces. On the other hand the algebras of cyclic operad 1 are exactly commutative Frobenious algebras.

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Recall that E_n -algebras are algebras over the operad of little n -disks. The first recognition principle for E_n -algebras was formulated by Stasheff :

- A topological space is a E_1 -algebra if and only if it is an algebra of a contractible nonsymmetric operad. (Stasheff 1964)

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- A topological space is an E_2 -algebras if it is an algebra over a contractible braided operad (Fiedorowicz 1992).

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Theorem (Batanin, 2004)

A topological space is a E_n -algebra if it is an algebra over a contractible n -operad.

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The proof is based on a geometric morphism:

$$des_n : SO(Top) \rightarrow O_n(Top)$$

which has left adjoint sym_n .

Here, $SO(Top)$ is the category of symmetric operads in Top and $O_n(Top)$ is the category of n -operads in Top .

We would like to calculate the left derived functor of symmetrisation $Lsym_n$.

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To compute the derived functor of symmetrisation we have to choose a cofibrant replacement for $\mathbb{1}$. One can show that geometric realization of H^T , where T is free n -operad monad on n -collections is such a replacement.

Moreover, $\text{sym}_n(|H^T|)$ is homotopy equivalent to the operad of Fulton-MacPherson compactification of the moduli space of configuration of points in \mathbb{R}^n , which is homotopy equivalent to the little n -disks operad.

Therefore, the category of E_n -algebras is Quillen equivalent to the category of $\text{sym}_n(|H^T|)$ -algebras. This proves the theorem.

Higher braided operads and stabilization hypothesis

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Stabilization hypothesis generalises classical Freudental stabilisation theorem.

- In *Set* one can consider only one level of commutativity for monoids.
- In *Cat* we have two levels: braided and symmetric monoidal categories.
- Breen-Baez-Dolan stabilization hypothesis predicts that in Cat_k we will have exactly $k + 1$ -levels of commutativity.

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Definition

An n -braided object in a symmetric monoidal model category V is an algebra of a contractible n -operad.

For $V = \mathit{Cat}_k$ an n -braided object is the same as n -braided k -category.

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Applications

A contractible n -operad is an example of an n -braided operad. If V is a monoidal model category then there exists a model category of n -braided operads $O_n^{loc}(V)$.

There is a geometric morphism

$$S^* : O_{n+1}^{loc}(V) \rightarrow O_n^{loc}(V).$$

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Applications

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There is a geometric morphism

$$S^* : O_{n+1}^{loc}(V) \rightarrow O_n^{loc}(V).$$

Again we are interested in calculation of the derived functor of left adjoint $S_!$.

Theorem (Batatin, Berger, Cisinski)

Let V be a symmetric h -monoidal model category which is k -truncated. Then there is a pair of Quillen equivalences $S_!, S^$ between $O_n^{loc}(V)$ and $O_{n+1}^{loc}(V)$ for $n \geq k + 2$.*

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by adjunction we have a corollary

Corollary

The category of n -braided objects in V is Quillen equivalent to the category of $n + 1$ -braided objects in V .

- For $V = Top_k$ (k -truncated homotopy type) it gives classical Freudental theorem.
- For $V = Cat_k$ (the category of k -category) this is the Breen-Baez-Dolan stabilization hypothesis.

Costello approach to open TCFT

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Applications

K. Costello (2004) introduced a moduli space \bar{N} of Riemann surfaces with boundary. A space $\bar{N}_{g,n,r}$ is the moduli space of Riemann surfaces of genus g with n boundary components and r marked points on the boundary. We also allow to have nodes on the boundary.

These spaces form a modular operad. The algebra of this modular operad is called open topological conformal field theory.

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Applications

Let $Mod : CycO(Top) \rightarrow ModO(Top)$ be the left adjoint to the forgetful functor from modular of cyclic operads.

Theorem (Costello 2004)

$$LMod(1) \simeq \bar{H}.$$

Corollary

The category of open topological conformal field theories is Quillen equivalent to the category of Frobenious A_∞ -algebras.

Corollary

The category of open topological field theories is equivalent to the category of Frobenious algebras.

Challenge

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Applications

- Construct a higher dimensional operad whose algebras are extended topological field theories.
- Construct a higher dimensional operad whose algebras are fully dualisable objects.
- Construct a geometric morphism between the categories of these operads.
- Prove Baez-Dolan cobordism hypothesis.