Dynamical invariants of foliated manifolds

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September 30, 2019

Motivating question

Can we probe the global behaviour of an integrable system by calculating path integrals?

Foliated manifolds

Let M be a manifold. A subbundle $E \subset TM$ is said to be *integrable* if $\Gamma^\infty(M; E)$ is closed under Lie brackets.

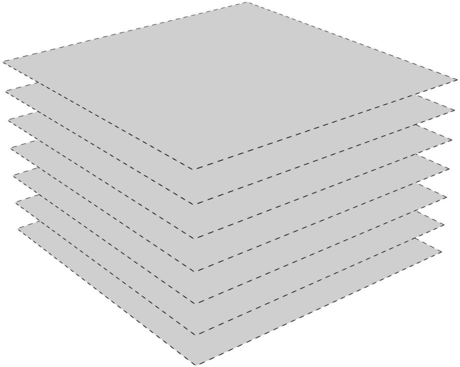
By the Frobenius theorem, an integrable subbundle E of TM *integrates* to a family

$$\mathcal{F} := \{(L_\lambda, i_\lambda : L_\lambda \hookrightarrow M)\}_{\lambda \in \Lambda}, \qquad M = \bigcup_{\lambda \in \Lambda} i_\lambda(L_\lambda)$$

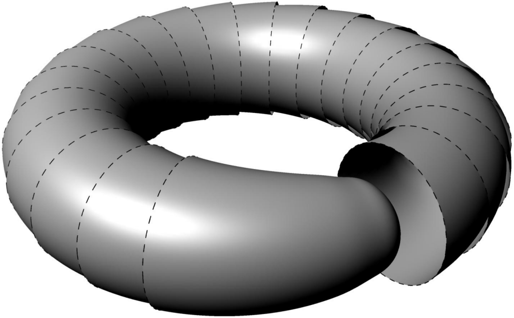
of M into disjoint, connected, immersed submanifolds called *leaves*. $E = T\mathcal{F}$, the tangent bundle to the leaves.

The pair $(M, \mathcal{F})$ is referred to as a *foliated manifold*. The rank p of $T\mathcal{F}$ is the *leaf dimension* of $\mathcal{F}$ and the rank $q = n - p$ of the normal bundle $N := TM/T\mathcal{F}$ is called the *codimension*.

Foliated manifolds



Foliated manifolds



The Godbillon-Vey invariant

Let M be equipped with a metric and let N be orientable. Then N identifies with a subbundle of M such that $TM = N \oplus T\mathcal{F}$. Define a connection $\nabla^\flat$ on N by

$$\nabla^\flat_X(Y) := [X_{T\mathcal{F}}, Y]_N + \left(\nabla^{LC}_{X_N} Y\right)_N.$$

The connection $\nabla^\flat$ is:

1. A *Bott connection*, in the sense that in any foliated coordinate system in which
$$\nabla^\flat = d + \alpha,$$
one has $\alpha|_{T\mathcal{F}} \equiv 0$. Globally, $\nabla^\flat$ is flat along leaves.
2. *Torsion-free*, in the sense that for all $X, Y \in \Gamma^\infty(M; N)$ one has
$$T^{\nabla^\flat}(X, Y) := \nabla^\flat_X(Y) - \nabla^\flat_Y(X) - [X, Y]_N \equiv 0.$$

The Godbillon-Vey invariant

The Riemannian metric g on M restricts to a Euclidean structure on N, whose determinant defines a trivialisation $\omega : \Lambda^q(N) \to M \times \mathbb{R}$.

The form $\omega \in \Gamma^\infty(M; \Lambda^q(N^*))$ extends by zero to $\Lambda^q(TM)$ to define a *transverse volume form* $\omega \in \Omega^q(M)$.

In the trivialisation of $\Lambda^q(N^*)$ determined by ω, the induced connection $(\nabla^\flat)^{(q)}$ on $\Lambda^q(N^*)$ has the form

$$(\nabla^\flat)^{(q)} = d + \eta$$

globally, for some $\eta \in \Omega^1(M)$. The curvature of $(\nabla^\flat)^{(q)}$ is $d\eta \in \Omega^2(M)$.

The Godbillon-Vey invariant

Since $\nabla^\flat$ is flat along leaves, Bott's vanishing theorem implies that

$$(d\eta)^{q+1} = 0$$

in $\Omega^*(M)$, so the form

$$\eta \wedge (d\eta)^q$$

is closed, defining a class

$$gv := [\eta \wedge (d\eta)^q] \in H^{2q+1}_{dR}(M).$$

The class gv is called the *Godbillon-Vey invariant* of $(M, \mathcal{F})$.

Characteristic classes

Let $F^+(N)$ denote positively oriented transverse frame bundle for N, and let $P := F^+(N)/\mathrm{SO}(q)$. P is called the *bundle of transverse metrics*.

The fibre $F^+(N)_x$ over $x \in M$ consists of all positively oriented linear isomorphisms $\phi : \mathbb{R}^q \to N_x$. The fibre P_x consists of classes $[\phi]$.

Any $[\phi] \in P_x$ determines a metric $\langle \cdot, \cdot \rangle_{[\phi]}$ on N_x defined by

$$\langle n_1, n_2 \rangle_{[\phi]} := \phi^{-1}(n_1) \cdot \phi^{-1}(n_2), \qquad n_1, n_2 \in N_x.$$

Thus a smooth section $\sigma : M \to P$ is the same thing as a smooth Euclidean structure for N.

Characteristic classes

Let $\alpha^\flat \in \Omega^1(F^+(N); \mathfrak{gl}(q,\mathbb{R}))$ and $R^\flat \in \Omega^2(F^+(N); \mathfrak{gl}(q,\mathbb{R}))$ be the connection and curvature forms of $\nabla^\flat$, and let

$$\alpha^\flat = \alpha^\flat_a + \alpha^\flat_s, \qquad R^\flat = R^\flat_a + R^\flat_s$$

be their decompositions into antisymmetric and symmetric components respectively.

Define $\underline{WO}_q := \Lambda(h_1, h_3, \dots, h_{q'}) \otimes \mathbb{R}[c_1, \dots, c_q]_q$. Define a differential d defined on generators by

$$dc_i = 0, \qquad \text{for all } i,$$

$$dh_i = c_i, \qquad \text{for } i \text{ odd.}$$

Then $\underline{WO}_q$ is a differential graded algebra.

Characteristic classes

Theorem (Bott-Guegorlet)

The formulae

$$c_i \mapsto \mathrm{Tr}\left((R^\flat)^i\right) \in \Omega^{2i}(P),$$

$$h_i \mapsto i \int_0^1 \mathrm{Tr}\left(\alpha_s^\flat(tR_s^\flat + R_a^\flat + (t^2-1)(\alpha_s^\flat))^{i-1}\right) dt \in \Omega^{2i-1}(P)$$

define homomorphisms $\phi_{\nabla^\flat} : \underline{WO}_q \to \Omega^*(P)$ *and* $\psi_{\nabla^\flat,\sigma} := \sigma^* \circ \phi_{\nabla^\flat}$, *and the the diagram*

$$\begin{array}{ccc} & & \Omega^*(P) \\ & \overset{\phi_{\nabla^\flat}}{\nearrow} & \\ \underline{WO}_q & & \big\downarrow \sigma^* \\ & \underset{\psi_{\nabla^\flat,\sigma}}{\searrow} & \\ & & \Omega^*(M) \end{array}$$

commutes. The induced maps on cohomology do not depend on $\nabla^\flat$ *or* σ.

Characteristic classes

In particular, the Godbillon-Vey invariant is represented in $\underline{WO}_q$ by $h_1 c_1^q$.

We have

$$\phi_{\nabla^\flat}(h_1 c_1^q) = \mathrm{Tr}(\alpha^\flat) \wedge \mathrm{Tr}(R^\flat)^q$$

and

$$\begin{aligned}\psi_{\nabla^\flat,\sigma}(h_1 c_1^q) &= \sigma^*\big(\mathrm{Tr}(\alpha^\flat) \wedge \mathrm{Tr}(R^\flat)^q\big) \\ &= \eta \wedge (d\eta)^q.\end{aligned}$$

The Godbillon-Vey invariant is closely related to the dynamics of $(M, \mathcal{F})$.

Holonomy

To study the dynamical behaviour of a general foliation $(M, \mathcal{F})$, we consider its *holonomy groupoid* $\mathcal{G}$.

An element of $\mathcal{G}$ is an equivalence class $u = [\gamma]$ of some smooth immersed path $\gamma : [0,1] \to M$ for which $\gamma'(t) \in T\mathcal{F}$ for all $t \in [0,1]$.

Two such paths γ_1 and γ_2 are deemed equivalent if they have the same endpoints and if their parallel transport maps $P_{\gamma_i}^{\nabla^\flat} : N_{s(\gamma_i)} \to N_{r(\gamma_i)}$ coincide for any Bott connection $\nabla^\flat$.

The set $\mathcal{G}$ of all such $u = [\gamma]$ can be equipped with a natural (non-Hausdorff) differential topology.

Holonomy

Let r and s be maps on $\mathcal{G}$ defined by

$$r([\gamma]) = \gamma(1),$$

$$s([\gamma]) = \gamma(0).$$

Then we have an inversion

$$[\gamma] \mapsto [\gamma]^{-1} := [\gamma^{-1}]$$

and a partially defined multiplication

$$[\gamma_1][\gamma_2] := [\gamma_1\gamma_2].$$

So $\mathcal{G}$ is a *Lie groupoid*. It is the *natural symmmetry object* associated to $(M, \mathcal{F})$, and it acts naturally (via parallel transport) on the vector bundle N and, therefore, on the bundle P.

Characteristic classes for $\mathcal{G}$

Cover $(M, \mathcal{F})$ with foliated charts U_i, and for each such U pick a local transversal T_i. Then $T := \bigcup_i T_i$ is a q-dimensional submanifold which intersects each leaf of $\mathcal{F}$, and which is everywhere transverse to $\mathcal{F}$.

One can then study $\mathcal{G}$ via the action of a (pseudo)group Γ on T. In these terms, we want a commuting diagram

$$\begin{array}{ccc} & & \Omega^*(P_T) \rtimes \Gamma \\ & \overset{\phi^\Gamma}{\nearrow} & \Big\downarrow \sigma^* \\ \underline{WO}_q & \underset{\psi^\Gamma}{\searrow} & \\ & & \Omega^*(T) \rtimes \Gamma. \end{array}$$

There have been a number of different attempts at such a construction.

The Connes-Moscovici map

Connes and Moscovici construct a characteristic map

$$H^*(\underline{WO}_q) \to HC^*(C_c^\infty(P_T) \rtimes \Gamma).$$

In particular they give a formula for the Godbillon-Vey invariant when $T = \mathbb{R}$. In this case $P_T \cong T \times \mathbb{R}_+^*$.

Theorem (Connes-Moscovici)

The Godbillon-Vey invariant is represented by the cyclic cocycle

$$gv(a^0, a^1) := \int_{\varphi \in \Gamma} a^0 \cdot (\varphi^* a^1) \cdot \frac{d}{dx}\big(\log \varphi'(x)\big) \cdot \frac{1}{t} dt \wedge dx$$

on $C_c^\infty(P_T) \rtimes \Gamma$.

The Connes-Moscovici formula comes from considering the trivial connection on $T = \mathbb{R}$. It hard to relate to the geometry of $(M, \mathcal{F})$.

A characteristic map for $\mathcal{G}$

It is more *geometrically meaningful* to work with the full holonomy groupoid $\mathcal{G}$ instead of the reduced $T \rtimes \Gamma$.

Conjecture

There exist explicit formulae, in terms of $\nabla^\flat$ and its curvature, for maps $\phi^{\mathcal{G}}_{\nabla^\flat}$, σ^* and $\psi^{\mathcal{G}}_{\nabla^\flat,\sigma}$ such that the diagram

$$
\begin{array}{ccc}
 & & \Omega^*(\mathcal{G}_P^{(*)}) \\
 & \overset{\phi^{\mathcal{G}}_{\nabla^\flat}}{\nearrow} & \Big\downarrow \sigma^* \\
\underline{WO}_q & & \\
 & \underset{\psi^{\mathcal{G}}_{\nabla^\flat,\sigma}}{\searrow} & \\
 & & \Omega^*(\mathcal{G}^{(*)})
\end{array}
$$

commutes.

A characteristic map for $\mathcal{G}$

Theorem (M.)

There exist explicit formulae for a characteristic map $\phi^{\mathcal{G}}_{\nabla^\flat} : \underline{WO}_q \to \Omega^*(\mathcal{G}^{(*)}_P)$ in terms of $\nabla^\flat$ and its curvature. The induced map on cohomology does not depend on $\nabla^\flat$.

When $(M, \mathcal{F})$ is codimension 1, the Godbillon-Vey form $\phi^{\mathcal{G}}_{\nabla^\flat}(h_1 c_1)$ on $\mathcal{G}$ can be used to construct the following cyclic cocycle.

Theorem (M.)

When $(M, \mathcal{F})$ is codimension 1, the formula

$$gv(a^0, a^1) := \int_{u_0 u_1 \in P} a^0(u_0) a^1(u_1) \frac{1}{t} dt \wedge R^{\mathcal{G}}_{u_1}$$

defines a cyclic cocycle on $C^\infty_c(\mathcal{G}_P)$.

A geometric interpretation

Recall that $u = [\gamma]$ is the *class of a leafwise path* γ *in* M.

Theorem (M.)

The differential form $R^{\mathcal{G}}$ *on* $\mathcal{G}$ *can be realised as a path integral of the Bott curvature* $R^\flat$. *More specifically*

$$R^{\mathcal{G}}_u = \int_\gamma R^\flat$$

where γ *is a representative of* u.

The Godbillon-Vey cyclic cocycle can now be written

$$gv(a^0, a^1) := \int_{\gamma_0\gamma_1 \in P} a^0(\gamma_0)a^1(\gamma_1)\frac{1}{t}dt \wedge \int_{\gamma_1} R^\flat$$

and is one step closer to the classical representative $\alpha^\flat \wedge R^\flat$.

Relationship with noncommutative index theory

The path integrated curvature $\int_\gamma R^\flat$ can be encoded in a semifinite spectral triple.

Theorem (M., Rennie)

There exists a semifinite spectral triple $(C_c^\infty(\mathcal{G}_P), \mathcal{H}, \mathcal{B})$ for $C_c^\infty(\mathcal{G}_P)$ whose Chern character is the Godbillon-Vey cyclic cocycle. The Chern character is computed using the semifinite local index formula of Carey-Gayral-Rennie-Sukochev.

The spectral triple $(C_c^\infty(\mathcal{G}_P), \mathcal{H}, \mathcal{B})$ is constructed using $\mathcal{G}$-equivariant KK-theory, and exhibits the Godbillon-Vey cyclic cocycle as a fundamentally *noncommutative* phenomenon. The idea of using equivariant KK-theory to access such data goes back to Connes.

Relationship with noncommutative index theory

1. $\mathcal{H} = L^2(H^* \rtimes \mathcal{G}; gv)$, where H^* is the (co)horizontal bundle over P determined by $\nabla^\flat$, and where gv is a $\mathcal{G}$-invariant differential form on H^* determined by $\alpha^\flat \wedge R^\flat$,
2. the action of $\mathcal{G}$ on H^* is by
$$\gamma \cdot h := P_\gamma^{\nabla^\flat} h + \int_\gamma R^\flat$$
3. the operator $\mathcal{B}$ is Clifford multiplication by the basepoint:
$$(\mathcal{B}\xi)(h) := h \cdot \xi(h), \qquad \xi \in C_c^\infty(H^*),\ h \in H^*.$$
4. $C_c^\infty(\mathcal{G}_P)$ acts on $\mathcal{H}$ by convolution, and
$$[\mathcal{B}, f]_\gamma = f(\gamma) \int_\gamma R^\flat.$$

The road ahead

The commuting diagram

$$
\begin{array}{ccc}
 & & \Omega^*(\mathcal{G}_P^{(*)}) \\
 & \overset{\phi^{\mathcal{G}}_{\nabla^\flat}}{\nearrow} & \downarrow \sigma^* \\
\underline{WO}_q & & \\
 & \underset{\psi^{\mathcal{G}}_{\nabla^\flat,\sigma}}{\searrow} & \\
 & & \Omega^*(\mathcal{G}^{(*)})
\end{array}
$$

remains incomplete, and has been so for at least 40 years.

Using the global-geometric object $\mathcal{G}$ instead of its étale versions, we have some geometric clues about how to complete it:

1. path integrals,
2. "parallel transport equivariant" KK-theory.

End

Thank you for your attention.