A localised equivariant index for proper actions

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Analysis on manifolds
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Joint work

Joint work with Hao Guo and Mathai Varghese:

- *Equivariant Callias index theory via coarse geometry*, ArXiv:1902.07391.
- *Coarse geometry and Callias quantisation*, ArXiv:1909.11815.

1. Localised indices

2. A localised equivariant index for proper actions

3. Maximal group C^*-algebras

I Localised indices

The setup

Throughout this talk,

- M is a complete Riemannian manifold;
- $E = E^+ \oplus E^- \to M$ is a $\mathbb{Z}_2$-graded, Hermitian vector bundle;
- D is an odd, elliptic, first order differential operator on E, self-adjoint on a domain in $L^2(E)$. We write

$$D^\pm := D|_{\Gamma^\infty(E^\pm)} : \Gamma^\infty(E^\pm) \to \Gamma^\infty(E^\mp).$$

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Consider the completion $W^1(E)$ of $\Gamma_c^\infty(E)$ in the norm

$$\|s\|_{W^1}^2 := \|s\|_{L^2}^2 + \|Ds\|_{L^2}^2.$$

Theorem (Anghel, 1993)

The following are equivalent:

- $D\colon W^1(E) \to L^2(E)$ *is Fredholm;*
- *there are a* $c > 0$ *and a compact set* $Z \subset M$ *such that for all* $s \in \Gamma_c^\infty(E)$ *supported outside* Z*,*

$$\boxed{\|Ds\|_{L^2} \geq c\|s\|_{L^2}.}$$

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In the setting of this theorem, we have the **localised index**

$$\mathrm{index}(D) := \dim\ker(D^+) - \dim\ker(D^-).$$

Special case 1: the Gromov–Lawson index

Suppose D is a Dirac-type operator, and

$$D^2 = \Delta + R,$$

where $R \in \mathrm{End}(E)$ is uniformly **positive outside a compact set**. Then D is Fredholm. (Gromov–Lawson, 1983.)

Special case 2: Callias operators

Suppose D is a **Callias operator**

$$D = D_0 + \Phi,$$

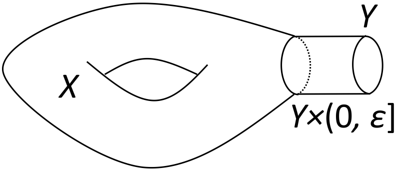
for a Dirac-type operator D_0, and $\Phi \in \mathrm{End}(E)$ such that

- $D_0\Phi + \Phi D_0$ is a bounded vector bundle endomorphism;
- $D_0\Phi + \Phi D_0 + \Phi^2$ is uniformly positive outside a compact set.

Then D is Fredholm.

Special case 3: Manifolds with boundary

Let X be a compact manifold with boundary Y. Suppose a neighbourhood U of Y is isometric to $Y \times (0, \varepsilon]$.



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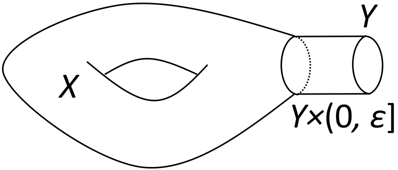


Suppose D_X is a Dirac-type operator on $E_X \to X$ such that

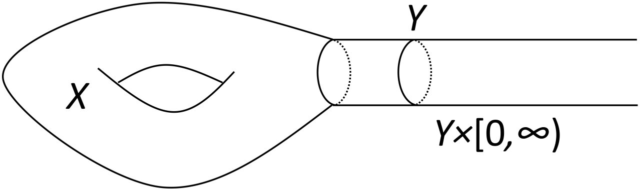
$$D_X|_U = \sigma\Big(D_Y + \frac{\partial}{\partial u}\Big),$$

where

- D_Y is a Dirac operator on $E_X^+|_Y \to Y$;
- $\sigma\colon E_X^+|_Y \xrightarrow{\cong} E_X^-|_Y$;
- u is the coordinate in $(0, \varepsilon]$.

Special case 3: Manifolds with boundary (cont'd)

Form M by glueing $Y \times [0, \infty)$ to X along U:



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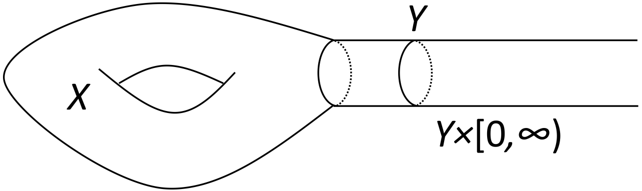

Let E and D be the extensions of E_X and D_X to M.

If D_Y is invertible, then D is Fredholm.

If D_Y is not invertible, we can still define a Fredholm operator by conjugating D by a weight function (i.e. using weighted Sobolev spaces).

Now index(D) is the **Atiyah–Patodi–Singer (APS) index** of D_X.

A localised equivariant index

Suppose D is invertible at infinity. If a compact group G acts on M and E, preserving D, then we have the **localised equivariant index**

$$\begin{aligned}\mathrm{index}_G(D) := [\ker(D^+)] - [\ker(D^-)] \\ \in R(G) := \{[V] - [W]; V, W \text{ fin. dim. reps}\}.\end{aligned}$$

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Goal: generalise this to general **locally compact** G and apply in various settings.

Issue: if G is noncompact and acts properly, commuting with D, then D will not be invertible outside a compact set, so not Fredholm.

II A localised equivariant index for proper actions

Proper actions

From now on, let G be a locally compact group, acting isometrically on M. Suppose the action is **proper**: the map

$$\begin{aligned} M \times G &\to M \times M \\ (m, g) &\mapsto (m, gm) \end{aligned}$$

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Suppose E is a G-equivariant vector bundle, and that G preserves D and the grading and metric on E.

We call a G-invariant subset $Z \subset M$ **cocompact** if Z/G is compact.

K-theory of group C^*-algebras

Definition

The **reduced group** C^* **algebra** of G is

$$C^*_r(G) := \overline{\{f * -; f \in L^1(G)\}} \quad \subset \mathcal{B}(L^2(G)).$$

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Definition

Let $\mathcal{H}$ be a Hilbert space, and $A \subset \mathcal{B}(\mathcal{H})$ a closed subalgebra, closed under adjoints, containing the identity. The **even K-theory** of A is

$$K_0(A) := \{[e_1] - [e_2]; e_j \in M_n(A) \text{ for some } n, e_j^2 = e_j\}.$$

(If A does not contain the identity, then elements of $K_0(A)$ are represented by formal differences of idempotents in the unitisation of A.)

The analytic assembly map

If M/G is compact, then D has an index, the **analytic assembly map**

$$\mathrm{index}_G(D) \in K_0(C_r^*(G)).$$

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The assembly map is the most commonly used index for proper, cocompact actions. It is used in the Baum–Connes conjecture, among other places.

A localised equivariant index for proper actions

Suppose that D is invertible outside a cocompact set.

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First goal: complete the table by constructing *

	G compact	G noncompact
M/G compact	Classical equivariant index	Analytic assembly map
M/G noncompact	Localised equivariant index	*

The localised Roe algebra

Let $Z \subset M$ be any closed subset.

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An operator $T \in \mathcal{B}(L^2(E))$

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3. is **supported near** Z if there is an $r > 0$ such that for all $f \in C_0(M)$ with support at least a distance r from Z, $fT = Tf = 0$.

Definition

The **Roe algebra of** M **for** E**, localised at** Z is

$$C^*(M; Z, E) := \overline{\{T \in \mathcal{B}(L^2(E)); 1\text{–}3 \text{ hold}\}} \quad \subset \mathcal{B}(L^2(E)).$$

Roe's localised coarse index

Let $Z \subset M$ be any closed subset. Suppose that there is a $c > 0$ such that for all $s \in \Gamma^\infty_c(E)$ supported outside Z, $\|Ds\|_{L^2} \geq c\|s\|_{L^2}$.

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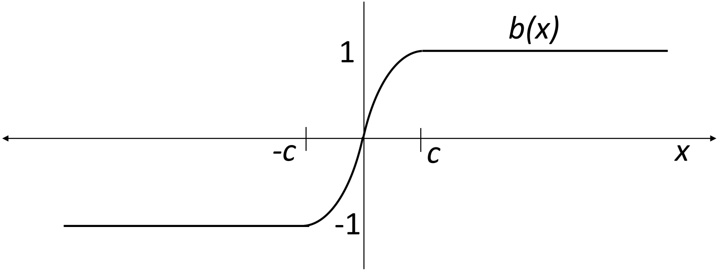
Let $b \in C^\infty(\mathbb{R})$ be odd and increasing, such that $b(x) = \pm 1$ if $|x| \geq c$.



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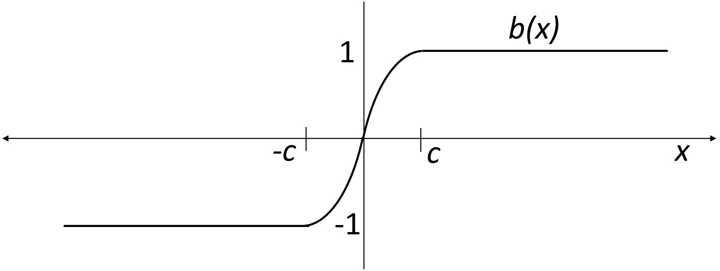

Using functional calculus, we form $b(D) \in \mathcal{B}(L^2(E))$.

Theorem (Roe, 2016)

1. *$b(D)$ is a multiplier of $C^*(M; Z, E)$;*
2. *$S := b(D)^2 - 1_E \in C^*(M; Z, E)$.*

Roe's localised coarse index (cont'd)

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So

$$e := \begin{pmatrix} (S^+)^2 & S^+(1_{E^+} + S^+)b(D)^- \\ S^- b(D)^+ & 1_{E^-} - (S^-)^2 \end{pmatrix}$$

lies in $C^*(M; Z, E)$, and is in fact an **idempotent**.

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Definition

The **localised coarse index** of D is

$$\mathrm{index}_Z(D) := [e] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1_{E^-} \end{pmatrix}\right] \quad \in K_0(C^*(M; Z, E)).$$

(Motivation: boundary map in 6-term exact sequence in K-theory.)

Special cases

If M is **compact**, then

- a locally compact operator on $L^2(E)$ is compact (take $f = 1$);
- every operator has finite propagation and is localised near any set (take $r > \mathrm{diam}(M)$),

so $C^*(M; Z, E) = \mathcal{K}(L^2(E))$ for any $Z \subset M$. Now index_Z is the usual index in

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More generally, if Z is compact, then

$$K_0(C^*(M; Z, E)) = K_0(C^*(Z; Z, E|_Z)) = \mathbb{Z};$$
$$\mathrm{index}_Z(D) = \mathrm{index}(D),$$

and we recover Anghel's localised index.

The equivariant localised Roe algebra

We now consider the proper action by the locally compact group G, as before. Suppose $Z \subset M$ is closed and G-invariant.

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The **equivariant Roe algebra of M for E, localised at Z** is

$$C^*(M; Z, E)^G := \overline{\{T \in \mathcal{B}(L^2(E) \otimes L^2(G))^G; 1\text{–}3 \text{ hold}\}} \subset \mathcal{B}(L^2(E) \otimes L^2(G))$$

The factor $L^2(G)$ is included to capture enough group-theoretic information. ($L^2(E) \otimes L^2(G)$ is an **admissible module**.)

Theorem (Guo–H–Mathai, 2019)

If G is unimodular and Z/G is compact, then $C^*(M; Z, E)^G \cong C^*_r(G) \otimes \mathcal{K}$.

This was known for discrete groups. Suppose from now on that G is unimodular.

The equivariant localised index

Suppose that there is a $c > 0$ such that for all $s \in \Gamma_c^\infty(E)$ supported outside a cocompact set Z, $\|Ds\|_{L^2} \geq c\|s\|_{L^2}$, and let $b \in C^\infty(\mathbb{R})$ be as before.

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Using an equivariant, isometric embedding $L^2(E) \hookrightarrow L^2(E) \otimes L^2(G)$, we view $b(D)$ as an operator on $L^2(E) \otimes L^2(G)$.

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The **localised equivariant coarse index** of D is

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The assumption that Z/G is compact is not essential, but implies that the index lands in a well-known and relevant K-theory group, and is independent of Z.

Special case 1: Callias operators

If $D = D_0 + \Phi$ is a Callias operator, Hao Guo constructed

$$\mathrm{index}_G^{\mathrm{Guo}}(D) \in K_0(C_r^*(G)).$$

Theorem (Guo–H–Mathai, 2019)

If D is a Callias operator,

$$\mathrm{index}_G(D) = \mathrm{index}_G^{\mathrm{Guo}}(D).$$

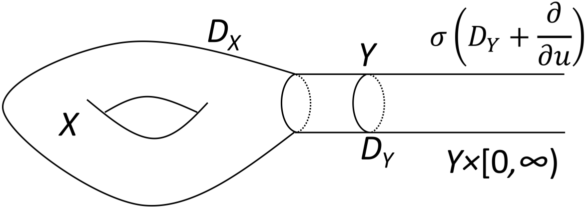
Corollary

If M/G is compact, $\mathrm{index}_G(D)$ is the image of D under the analytic assembly map.

So the index indeed fits into the table a few slides back.

Special case 2: Manifolds with boundary

Consider the case of a proper, isometric G-manifold X with boundary Y, now with X/G compact.



Suppose that D_Y is invertible. Then we have the **equivariant APS index**

$$\text{index}_G(D) \in K_0(C^*_r(G)).$$

This generalises to the case where 0 is isolated in the spectrum of D_Y.

See Hang Wang's talk tomorrow for an APS-type index theorem for this index.

III Maximal group C^*-algebras

The maximal group C^*-algebra

The **maximal group C^*-algebra** of G is the completion of the convolution algebra $L^1(G)$ in the norm

$$\|f\|_{\max} := \sup_{\pi \in \hat{G}} \|\pi(f)\|.$$

For all $f \in L^1(G)$,

$$\|f * -\|_{\mathcal{B}(L^2(G))} \leq \|f\|_{\max} \leq \|f\|_{L^1(G)}.$$

In particular, the supremum is finite.

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Taking π to be the trivial representation 1_G, we find that

$$\Big|\int_G f(g) dg\Big| = \|1_G(f)\| \leq \|f\|_{\max}.$$

So integrating functions in $L^1(G)$ extends continuously to the **integration trace** $I \colon C^*_{\max}(G) \to \mathbb{C}$.

The invariant index for cocompact actions

If M/G is compact, then the analytic assembly map for the maximal group C^*-algebra gives

$$\mathrm{index}_G(D) \in K_0(C^*_{\max}(G)).$$

The integration trace induces

$$I_*\colon K_0(C^*_{\max}(G)) \to K_0(\mathbb{C}) = \mathbb{Z}.$$

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Theorem (Bunke–Mathai–Zhang, 2010)

If M/G is compact and G is unimodular, then

$$I_*(\mathrm{index}_G(D)) = \dim(\ker(D^+))^G - \dim(\ker(D^-))^G.$$

Mathai and Zhang proved a **quantisation commutes with reduction** result in terms of this index.

Refining non-equivariant indices

Suppose that M is the universal cover of a compact manifold X, and D is the lift of an elliptic operator D_X on X. Let $G := \pi_1(X)$. Then

$$I_*(\mathrm{index}_G(D)) = \dim(\ker(D^+))^G - \dim(\ker(D^-))^G$$

becomes

$$I_*(\mathrm{index}_{\pi_1(X)}(D)) = \mathrm{index}(D_X).$$

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In this sense, $\mathrm{index}_{\pi_1(X)}(D)$ **refines** $\mathrm{index}(D_X)$.

In particular, if D_X is a Spin-Dirac operator, then

$$\mathrm{index}_{\pi_1(X)}(D) \in K_0(C^*_{\max}(\pi_1(X)))$$

is a stronger obstruction to Riemannian metrics of positive scalar curvature than $\hat{A}(X) = \mathrm{index}(D_X) \in \mathbb{Z}$.

A maximal localised index

Goals:

1. develop a **maximal equivariant localised index**

$$\mathrm{index}_G(D) \in K_0(C^*_{\max}(G))$$

   if M/G is noncompact;

2. prove that $I_*(\mathrm{index}_G(D)) \in \mathbb{Z}$ equals a concrete index in terms of G-invariant sections;
3. apply this to (for example) positive scalar curvature and geometric quantisation.

A maximal localised Roe algebra

Let $C^*_{\mathrm{alg}}(M, E)$ be the algebra of locally compact, finite propagation operators on $L^2(E)$.

Lemma (Gong–Wang–Yu, 2008)

The maximal Roe algebra norm

$$\|T\|_{\max} := \sup_{\pi} \|\pi(T)\|,$$

is **finite** *for all* $T \in C^*_{\mathrm{alg}}(M, E)$*, where the supremum is over all* $*$*-representations of* $C^*_{\mathrm{alg}}(M, E)$*.*

Completing $C^*_{\mathrm{alg}}(M, E)$ in this norm, we obtain the **maximal Roe algebra** $C^*_{\max}(M, E)$.

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Completing $C^*_{\mathrm{alg}}(M, E)$ in this norm, we obtain the **maximal Roe algebra** $C^*_{\max}(M, E)$.

It is unclear a priori if this extends to the general equivariant setting. But if Z/G is compact, then $C^*_{\mathrm{alg}}(M; Z, E)^G$ is a dense subalgebra of $L^1(G) \otimes \mathcal{K}$. So it has a maximal completion $C^*_{\max}(M; Z, E)^G \cong C^*_{\max}(G) \otimes \mathcal{K}$.

Operators on Hilbert C^*-modules

The definition of the non-maximal localised coarse index was based on this theorem.

Theorem (Roe, 2016)

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This does **not** extend to the maximal norm in an obvious way.

Solution: view D as an unbounded operator on the Hilbert $C^*_{\max}(M; Z, E)^G$-module $C^*_{\max}(M; Z, E)^G$ by composition, and use functional calculus on Hilbert C^*-modules.

Any C^*-algebra A is a Hilbert A-module, with A-valued inner product

$$(a, b)_A := a^* b.$$

Localising operators on Hilbert C^*-modules

Theorem (Guo–H–Mathai, 2019)

1. *The operator D on $C^*_{\max}(M; Z, E)^G$ is regular and essentially self-adjoint, so functional calculus applies.*
2. *With b as before, $b(D)^2 - 1 \in C^*_{\max}(M; Z, E)^G$.*

This allows us to define the **maximal equivariant localised index**

$$\mathrm{index}_G(D) \in K_0(C^*_{\max}(M; Z, E)^G) = K_0(C^*_{\max}(G)).$$

The invariant index

Let $\chi \in C(M)$ be such that for all $m \in M$,

$$\int_G \chi(gm)^2 \, dg = 1.$$

Definition

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Theorem (Guo–H–Mathai, 2019)

$$I_*(\mathrm{index}_G(D)) = \dim(\ker(D) \cap L^2_T(E^+)^G) - \dim(\ker(D) \cap L^2_T(E^-)^G).$$

Refining localised indices

Suppose that M is the universal cover of a noncompact manifold X, and D is the lift of an elliptic operator D_X on X that is invertible at infinity. Let $G := \pi_1(X)$. Then

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So as in the case where X is compact, $\mathrm{index}_{\pi_1(X)}(D)$ **refines** $\mathrm{index}(D_X)$.

In particular, if D_X is a (Callias) Spin-Dirac operator, then

$$\mathrm{index}_{\pi_1(X)}(D) \in K_0(C^*_{\max}(\pi_1(X)))$$

is a stronger obstruction to Riemannian metrics of positive scalar curvature than the Gromov–Lawson or Callias index of D_X.

Callias quantisation and reduction

Now suppose $D = D_0 + \Phi$ is a Callias operator, and D_0 is a Spin^c-Dirac operator. Suppose that G is a unimodular Lie group. There is a Spin^c-**moment map**

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The **reduced space** at 0 is

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Theorem (Guo–H–Mathai, 2019)

If M_0 is smooth, then for high enough powers of the determinant line bundle,

$$I_*(\mathrm{index}_G(D_0 + \Phi)) = \mathrm{index}(D_{M_0}),$$

for a Spin^c*-Dirac operator D_{M_0} on M_0.*

Thank you