KO valued spectral flow

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Overview

Joint work with Chris Bourne, Matthias Lesch, John Phillips, Adam Rennie, Hermann Schulz-Baldes.

The real case, physics

Quantum mechanics is formulated on a complex Hilbert space.

Wigner'(1930's) showed that symmetries in quantum mechanics can be implemented by conjugate linear operators (anti-unitary operators). These may be studied by regarding the complex Hilbert space as two copies of a real Hilbert space using a distinguished complex conjugation compatible with the symmetries.

The most complete discussion of such a situation is by Araki (60s) who looked at Majorana fermions (using a many body approach via his theory of 'self dual CAR algebras').

Araki showed how to reformulate the dynamics in the complex Hilbert space in terms of operators on the underlying real Hilbert space.

There it is given by a one parameter group $t \to e^{Ht}, t \in \mathbb{R}$ with H skew adjoint but commuting with the complex conjugation.

So Hamiltonians in this picture are skew adjoint and we therefore study skew adjoint Fredholm operators on a real Hilbert space.

The real case mathematically

Next, we note that on a real Hilbert space the analogue of the self adjoint Fredholms is the space of skew adjoint Fredholms.

This was all explained in a 60's IHES paper of Atiyah-Singer ("Index theory for skew-adjoint Fredholm operators"). Their paper is about the classifying spaces for KO theory in operator theory terms.

Today we appreciate that their construction (with hindsight) is a case of Kasparov's (1981) bivariant K-theory.

In particular they established the 8-fold periodicity of these classifying spaces for KO using homotopy theory and Clifford periodicity in the real case. They identified the homotopy groups of the classifying spaces and much more.

Notation: $\mathcal{H}_{\mathbb{R}}$ is a real separable Hilbert space, $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$ denotes the bounded operators on $\mathcal{H}_{\mathbb{R}}$ and $\mathcal{K}(\mathcal{H}_{\mathbb{R}})$ the compact operators.

Use $\hat{\mathcal{F}}^k(\mathcal{H}_{\mathbb{R}})$, $k = 0, 1, \cdots, 7$, for the classifying spaces of KO theory as defined in Atiyah-Singer.

Here $\hat{\mathcal{F}}^0(\mathcal{H}_{\mathbb{R}})$ denotes the Fredholm operators on a real Hilbert space.

The real skew adjoint Fredholms in the list above are denoted $\hat{\mathcal{F}}^1(\mathcal{H}_{\mathbb{R}})$.

The spaces $\hat{\mathcal{F}}^k(\mathcal{H}_{\mathbb{R}})$ for $k > 1$ are defined as the subspaces of $\hat{\mathcal{F}}^1(\mathcal{H}_{\mathbb{R}})$ that graded commute with a representation of the real Clifford algebra on $k - 1$ generators.

For a complex Hilbert space $\mathcal{H}$ we only have $\hat{\mathcal{F}}^k(\mathcal{H})$, $k = 0, 1$ with $\hat{\mathcal{F}}^1(\mathcal{H})$ being the self adjoint Fredholm operators.

In physical models the Clifford generators arise from symmetries of the system (time reversal and particle-hole symmetry) as explained by Guo Thiang (Adelaide) in 2014.

The space $\hat{\mathcal{F}}^1(\mathcal{H}_{\mathbb{R}})$ has the following homotopy groups.

First of all, π_0 is $\mathbb{Z}_2$, reflecting the two connected components distinguished by the original mod 2 index of Atiyah-SInger (1969): if F is real skew-adjoint then the mod 2 index is $\dim\ker F$ (mod 2).

A second group that has come up in applications is π_1, which is also $\mathbb{Z}_2$. This group is detected by 'real' or, '$\mathbb{Z}_2$' spectral flow just as π_1 of the self adjoint Fredholm operators on a complex Hilbert space is detected by spectral flow in the usual sense.

The other homotopy groups $\pi_2, \pi_4, \pi_5, \pi_6$ are all zero while $\pi_3 = \mathbb{Z} = \pi_7$.

These groups also arise as a Clifford module valued index of skew adjoint Fredholm operators. There is such an index for each of the classifying spaces as is explained in Atiyah-Singer drawing on the older paper of Atiyah-Bott-Shapiro.

In the physics literature on ‘topological insulators’ there has been a lot of discussion about $\mathbb{Z}_2$ indices recently. But one needs to be a bit careful because as we see above there are two $\mathbb{Z}_2$ homotopy groups and they have different interpretations from the classifying space point of view. (Also it is unclear whether the discussions are mathematically sound).

For example, Witten in 1986 gave an intuitive discussion of spectral flow in the real skew adjoint Fredholms. His idea is to use the same intersection number approach in the real case that was used by Atiyah-Patodi-Singer in the complex case. In 1988 John Lott extended Witten’s ideas to the other classifying spaces of Atiyah-Singer.

But: there are mistakes in both papers. Moreover the precise ‘intersection number’ definition along the lines of the treatment by Ruget (’85) and Getzler (’93) for the complex case requires a lot more work.

The applications to physical theory need an analytic approach as the topological approach advocated by Witten can only be exploited in very simple examples.

Hermann Schulz-Baldes and collaborators have been using analytic spectral flow methods for some time. The starting point for them was John Phillips 1995 paper in the complex case and a recent joint paper with me in the real case.

Our joint paper does not apply to the case where the Hamiltonian admits symmetries.

This talk is partly about an even more recent paper (now on the arXiv) intended to resolve this issue by introducing what we call 'KO-valued spectral flow'.

Our approach is based on an axiomatic characterisation of this notion of KO- spectral flow. The basic axioms are the three that apply in the complex case:

(i) normalisation, (that is, it gives the right answer for classical Toeplitz operators)
(ii) concatenation, (or additivity under the operation of joining two paths)
(iii) homotopy invariance.

plus we need a stability axiom because, in the real case, to obtain uniqueness we start from the finite dimensional situation.

In this talk I will sketch an argument for $\hat{\mathcal{F}}^1(\mathcal{H}_{\mathbb{R}})$ as for this case we can use a bare hands approach.

We consider the space of all complex structures on a real Hilbert space.

Any pair of complex structures J_1, J_2 are related by an orthogonal operator $J_2 = OJ_1O^T$. If we assume that the commutator $[J, O]$ is compact then we see that the operator $J + OJO^T$ is skew adjoint and Fredholm.

It is the real analogue of a generalised Toeplitz operator and it has a mod 2 index.

In fact the kernel of this operator is the bit of the Hilbert space where O and J anti-commute.

We introduce the group $\mathcal{O}$ of all orthogonal operators that have compact commutator with a fixed complex structure and we showed that it is a classifying space for KO.

Namely, it has the homotopy type of $\hat{\mathcal{F}}^2(\mathcal{H}_{\mathbb{R}})$.

The map that sends an operator O to the complex dimension mod 2 of the kernel of $J + OJO^T$ is a homomorphism of this group $\mathcal{O}$ onto $\mathbb{Z}_2$.

We know from earlier work that this map separates the connected components of $\mathcal{O}$.

We need this map later.

KO spectral flow

Skew adjoint Fredholm operators have a polar decomposition $K = J|K|$ where J is a skew adjoint real partial isometry (i.e. it may have a kernel).

By the Atiyah-Singer mod 2 index result for skew adjoint Fredholm operators there are two connected components distinguished by the dimension (mod 2) of the kernel.

We focus on the component having mod 2 index zero (even dimensional kernel).

Any skew adjoint real partial isometry with even dimensional kernel can be turned into a complex structure by taking a direct sum with a complex structure defined only on its kernel.

We showed there is no dependence of our definition of KO flow on the complex structure on the kernel.

Thus let J_1 and J_2 be two complex structures that are 'close' in the Calkin algebra, $\mathcal{Q}$, in the sense that $||J_1 - J_2||_{\mathcal{Q}} < 1$.

Then we may join them by a straight line path that stays in the space of skew adjoint Fredholms.

This is because each operator in the path is norm close to one of the endpoints in $\mathcal{Q}$ and hence invertible in $\mathcal{Q}$.

We define the spectral flow along this path in terms of the kernel dimension of $J_1 + J_2$.

But we need to be careful here.

On the kernel we have $J_1 = -J_2$ so we may equip the kernel with a complex structure defined by J_1 say.

Then the spectral flow $sf(J_1, J_2)$ along our path is defined to be the complex dimension of the kernel of $J_1 + J_2$.

This normalisation is correct because in terms of the axioms, the real Toeplitz operators fix a normalisation (as Toeplitz operators do in the complex case).

To obtain the KO-valued flow we assume that the pair J_1, J_2 lie in $\hat{\mathcal{F}}^k(\mathcal{H}_{\mathbb{R}})$ for $k > 1$. In this case we use the finite dimensional case to check the normalisation.

This can be deduced from Milnor's book on Morse theory where he uses a Clifford algebra approach to Bott periodicity.

In the $\hat{\mathcal{F}}^k(\mathcal{H}_{\mathbb{R}})$ case the kernel of $J_1 + J_2$ is a Clifford module for the real Clifford algebra $\mathcal{C}_{k-1}$ on $k - 1$ generators. The ABS Clifford index applies here.

We define the spectral flow sf_k along the straight line path joining J_1, J_2 as the ABS index of $\ker(J_1 + J_2)$.

Recall that the ABS construction is to regard the index as taking its values in the $\mathcal{C}_{k-1}$ modules modulo those that extend to be $\mathcal{C}_k$ modules.

The next step is to see that if we have a general norm continuous path $\{K_t\},\ t \in [0,1]$ in the skew adjoint Fredholms then we obtain by polar decomposition a (generally discontinuous) path of complex structures $\{J_t\}$.

But, in the Calkin algebra the path is continuous. We may thus divide the path into sub-paths $[t_j, t_{j+1}]$ such that the corresponding pairs of complex structures $J_{t_j}, J_{t_{j+1}}$ (and all the intermediate ones) are close in the Calkin algebra.

We then define the spectral flow along the path to be the sum

$$\sum_j sf_k(J_j, J_{j+1}).$$

One may check that this coincides with the intersection number point of view by complexifying and comparing with Phillips approach using the index of Fredholm pairs of projections.

That this is independent of the subdivision may be proved in several ways.

Spectral flow is then easily seen to be additive on paths (concatenation).

Homotopy invariance also follows in a straightforward fashion.

For example, in our original approach to obtaining the isomorphism on the fundamental group of the skew adjoint Fredholms we introduced a map from the skew adjoint Fredholms to the group $\mathcal{O}$.

Recall that the latter has the homotopy type of $\hat{\mathcal{F}}^2(\mathcal{H}_\mathbb{R})$ and we used the previously mentioned fact that there is an isomorphism of π_0 of this space with $\mathbb{Z}_2$. From this we may prove that spectral flow around loops gives an isomomorphism to $\mathbb{Z}_2$.

For *KO*-valued flow we used a different argument.

Application

Schulz-Baldes and I considered some model Hamiltonians (in the so-called tight binding approximation) that that are time reversal symmetric. They led to skew adjoint Fredholm operators in $\hat{\mathcal{F}}^1(\mathcal{H})$.

These arise from the one dimensional Kitaev model and some Bogoliubov-de Gennes Hamiltonians.

By perturbing the Hamiltonian $H \to H + V$ we asked whether there is any spectral flow along the straight line path joining them for certain choices of perturbation V.

In these examples we find non-zero spectral flow.

Bourne and Schulz-Baldes recently studied spin chains including some with interaction and found similar results.

With his student Nora Doll, Schulz-Baldes introduced the notion of 'parity flow'. This turns out to fit into our KO-valued spectral flow providing a further example.

Our results may also be applied to the framework of Allwood-Max-Zirnbauer.

In that context I should mention that everything can be done for Clifford algebras $\mathcal{C}_{r,s}$.

In summary our KO valued flow encompasses all previous instances of spectral flow that have been studied in a mathematically precise fashion.

The unbounded case

In applications to physics we need to allow the Hamiltonian to be unbounded.

There are various options for defining unbounded Fredholm operators and for the purposes of this talk I will follow Atiyah-Singer.

They use the Riesz map and on unbounded skew adjoint operators this is : $D \mapsto D(1 - D^2)^{-1/2}$.

We say D is Fredholm if its image under this map is Fredholm.

The KO flow can then be defined for skew-adjoint unbounded operators.

The real Robbin-Salamon theorem

Theorem

Let $A(\cdot)$ be a family of unbounded skew-adjoint Fredholm operators in $\hat{\mathcal{F}}^k(\mathcal{H}_\mathbb{R})$ for $k \geq 1$. Let

$$D = \begin{pmatrix} -\frac{d}{dt} & -A(t) \\ -A(t) & \frac{d}{dt} \end{pmatrix}. \tag{1}$$

Then D is an essentially skew-adjoint Fredholm operator on $L^2(\mathbb{R}, \mathcal{H}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R})$ whose bounded transform lies in $\hat{\mathcal{F}}^{k+1}(\mathcal{H}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R})$ for $k \geq 2$. With the previous normalizations we have

$$\mathrm{sf}_k(A(\cdot)) = \mathrm{ind}_{k+1}(D).$$

By introducing the 2×2 matrix of operators we have allowed an additional Clifford generator that anticommutes with D to be introduced, namely:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Lott proved a special case in his 1988 paper by following the argument in APS.

The proof of the theorem uses our axiomatic characterisation of spectral flow. Essentially what we do is show that mapping that goes via the RHS gives a Clifford index with the same properties as the map we already have on the LHS.

Work in progress.

We have generalised the notion of $KO \equiv KO(pt)$ valued spectral flow to $KO(A)$ valued spectral flow for A a real C^*=algebra.

One sees then a connection between the previous theorem and Kasparov's bivariant KK-theory in the real case.

At this time there does not seem to be a direct application of this more refined notion of spectral flow.