Wess-Zumino-Witten Models on Lie Supergroups

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Wess-Zumino-Witten Models

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Wess-Zumino-Witten Models

WZW models describe (closed) strings on a Lie group G:

$$g : \Sigma \longrightarrow G.$$

Here, Σ is the worldsheet (a Riemann surface) and G is the target space (a connected Lie group). $g^{-1}\mathrm{d}g = g^*\vartheta$ is the pullback of the canonical ($\mathfrak{g}$-valued) 1-form to Σ. The WZW action is:

$$S_{\mathrm{WZW}} = \frac{k}{8\pi} \int_\Sigma \kappa\left(g^{-1}\mathrm{d}g, \star\left(g^{-1}\mathrm{d}g\right)\right) \\ - \frac{k}{2\pi\mathfrak{i}} \int_\Gamma \kappa\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}, \mathrm{d}\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\right)\right).$$

Here, κ is the Killing form (or something similar), $\star$ is the Hodge star on Σ, $\partial\Gamma = \Sigma$ and $\tilde{g} : \Gamma \to G$ extends g.

Classical Equations of Motion

The EOMs turn out to be

$$\overline{\partial} J(z) = \partial \overline{J}(\overline{z}) = 0,$$

where

$$J(z) = k g^{-1} \partial g, \qquad \overline{J}(\overline{z}) = -k \overline{\partial} g g^{-1},$$

and $(z, \overline{z})$ are local complex coordinates on Σ.

Because the 1-forms $J(z)$ and $\overline{J}(\overline{z})$ are $\mathfrak{g}$-valued, we decompose along a basis T_a of $\mathfrak{g}$:

$$J(z) = \sum_a J^a(z) \otimes T_a, \qquad \overline{J}(\overline{z}) = \sum_a \overline{J}^a(\overline{z}) \otimes T_a.$$

[From now on, all holomorphic results have an antiholomorphic partner!]

Quantisation

For path-integral quantisation, we need the (euclidean) Feynman amplitudes $e^{-S_{WZW}}$ to be well-defined. This *may* constrain the coupling constant k, depending on $H^3(G;\mathbb{Z})$.

eg. G compact, simply-connected and semisimple means $H^3(G;\mathbb{Z}) \cong \mathbb{Z}$ and k is constrained to be an integer.

Canonical quantisation of the Fourier modes of the fields

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

results in the untwisted affine Kac-Moody algebra $\widehat{\mathfrak{g}}$:

$$\left[J^a_m, J^b_n\right] = \left[J^a, J^b\right]_{m+n} + m\kappa\left(J^a, J^b\right)\delta_{m+n,0}k.$$

The space $\mathcal{H}$ of quantum states is a $\widehat{\mathfrak{g}}$-module.

CFTs and VOAs

The Sugawara construction defines modes

$$L_n = \frac{1}{2\left(k + \mathsf{h}^\vee\right)} \sum_{a,b} \kappa^{-1}\left(J^a, J^b\right) \sum_{r\in\mathbb{Z}} :J^a_r J^b_{n-r}: ,$$

where $\mathsf{h}^\vee$ is the dual Coxeter number and $:\cdots:$ denotes normal ordering. These give a copy of the Virasoro algebra

$$\left[L_m, L_n\right] = (m-n)\, L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\, c,$$
$$\left[L_m, J^a_n\right] = -nJ^a_{m+n}, \quad c = \frac{k \dim \mathfrak{g}}{k + \mathsf{h}^\vee}.$$

and so we get a conformal field theory or vertex operator algebra.

When G is compact, simply-connected and semisimple, $k \in \mathbb{N}$ and the VOA-reps are the integrable $\widehat{\mathfrak{g}}$-modules. There are only finitely many integrable modules and their category is semisimple.

Example: $\mathsf{G} = \mathsf{GL}(1)$ or $\mathsf{U}(1)$

Since $\mathsf{H}^3(\mathsf{GL}(1)) = \mathsf{H}^3(\mathsf{U}(1)) = 0$, there is no constraint on k. The CFT/VOA structure arises from $\widehat{\mathfrak{gl}}(1) = \widehat{\mathfrak{u}}(1)$:

$$\left[a_m, a_n\right] = m\delta_{m+n,0}k, \quad L_n = \frac{1}{2k}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}: , \quad c = 1.$$

Bounding the energy of the quantum states below leads us to highest weight states:

$$a_0|\lambda\rangle = \lambda|\lambda\rangle, \qquad a_n|\lambda\rangle = 0 \quad (n > 0).$$

Such a state spans a rep for a Borel subalgebra $\mathfrak{b}$ from which we construct Verma modules V_λ:

$$V_\lambda = \mathsf{Ind}_{\mathfrak{b}}^{\widehat{\mathfrak{u}}(1)} \mathbb{C}|\lambda\rangle.$$

Verma modules are irreducible!

Characters and Partition Functions

Given that every Verma module defines an irreducible VOA-rep, we expect that the quantum state space has the form

$$\mathcal{H} = \int_{\mathbb{R}} V_\lambda \otimes V_\lambda \, \mathrm{d}\lambda.$$

The character of a module is the function (write $q = \mathrm{e}^{2\pi \mathrm{i} \tau}$)

$$\chi_{V_\lambda}(q) = \mathrm{tr}_{V_\lambda} q^{L_0 - c/24} = \frac{\mathrm{e}^{\mathrm{i}\pi\lambda^2\tau}}{\eta(q)}, \quad \eta(q) = q^{1/24} \prod_{i=1}^{\infty} (1 - q^i),$$

which records the energy of the quantum states (on the cylinder). The partition function is the character of the quantum state space:

$$Z[q] = \int_{\mathbb{R}} \left| \chi_{V_\lambda}(q) \right|^2 \mathrm{d}\lambda = \frac{\int_{\mathbb{R}} \left| \mathrm{e}^{\mathrm{i}\pi\lambda^2\tau} \right|^2 \mathrm{d}\lambda}{\left| \eta(q) \right|^2} = \frac{1}{\sqrt{2 \,\mathrm{Im}\, \tau} \left| \eta(q) \right|^2}.$$

Consistency Conditions 1

For the CFT to be well-defined when Σ is a torus, the partition function must be a modular function — it must be invariant under $\mathsf{SL}(2;\mathbb{Z})$, *ie.* the S- and T-transformations

$$\mathsf{S}\colon \tau \longmapsto -\frac{1}{\tau}, \qquad \mathsf{T}\colon \tau \longmapsto \tau + 1.$$

Using $\eta(-1/\tau) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau)$ and $\eta(\tau+1) = \mathrm{e}^{\mathrm{i}\pi/12}\eta(\tau)$, we check:

$$\begin{aligned} S\colon \qquad & \sqrt{\operatorname{Im}\tau} \longmapsto \frac{\sqrt{\operatorname{Im}\tau}}{|\tau|} \qquad & |\eta(q)|^2 \longmapsto |\tau|\,|\eta(q)|^2, \\ T\colon \qquad & \sqrt{\operatorname{Im}\tau} \longmapsto \sqrt{\operatorname{Im}\tau} \qquad & |\eta(q)|^2 \longmapsto |\eta(q)|^2. \end{aligned}$$

Thus, $Z[q] = \dfrac{1}{\sqrt{2\operatorname{Im}\tau}\,|\eta(q)|^2}$ is modular invariant.

Consistency Conditions 2

The S-transformation of the Verma module characters may be written in the form

$$\chi_{V_\lambda}(-1/\tau) = \int_{\mathbb{R}} \mathsf{S}_{\lambda\mu} \chi_{V_\mu}(\tau)\, \mathrm{d}\mu, \qquad \mathsf{S}_{\lambda\mu} = \mathsf{e}^{2\pi\mathsf{i}\lambda\mu}.$$

A second consistency check is that the Verlinde formula gives non-negative structure constants:

$$\mathbf{N}_{\lambda\mu}{}^{\nu} = \int_{\mathbb{R}} \frac{\mathsf{S}_{\lambda\sigma}\mathsf{S}_{\mu\sigma}\mathsf{S}^*_{\sigma\nu}}{\mathsf{S}_{0\sigma}}\, \mathrm{d}\sigma = \int_{\mathbb{R}} \mathsf{e}^{2\pi\mathsf{i}(\lambda+\mu-\nu)\sigma}\, \mathrm{d}\sigma = \delta(\lambda+\mu-\nu)$$

$$\Rightarrow \qquad V_\lambda \times V_\mu = \int_{\mathbb{R}} \mathbf{N}_{\lambda\mu}{}^{\nu} V_\nu \, \mathrm{d}\nu = V_{\lambda+\mu}.$$

These are, in fact, the dimensions of certain spaces of “conformal blocks”!

$U(1)$: The Compactification

The WZW model on $GL(1)$ has:

- A continuous spectrum.
- Completely reducible modules.
- A modular-invariant partition function.
- A continuum Verlinde formula.

What happens when we replace $GL(1)$ by $U(1)$?

- String theorists: Compactification leads to momentum quantisation, hence a discrete spectrum.
- Geometers: Highest weight states correspond to global sections of a certain line bundle, hence the periodic boundary condition picks out a discrete spectrum.
- Algebraists: The algebra (VOA) is extended by a simple current whose untwisted modules give rise to a discrete spectrum.

Algebraic Compactification

We have seen the fusion rules $V_\lambda \times V_\mu = V_{\lambda+\mu}$ which imply that every module is invertible in the fusion ring.

The VOA corresponding to $\widehat{\mathfrak{gl}}(1) = \widehat{\mathfrak{u}}(1)$ may be extended by invertible elements:

$$\mathfrak{W}_\lambda \sim \bigoplus_{j \in \mathbb{Z}} V_{j\lambda} \qquad (\widehat{\mathfrak{u}}(1) \sim V_0).$$

The irreducible modules of the extended algebra have the form

$$W_{[\mu]} \sim \bigoplus_{j \in \mathbb{Z}} V_{\mu+j\lambda} \qquad (\mu \in \mathbb{R}/\lambda\mathbb{R}).$$

These are untwisted when the extended algebra acts with integer modes (the energies of the states all differ by integers). Need:

$$\lambda^2 \in \mathbb{Z}, \qquad \lambda\mu \in \mathbb{Z}.$$

Consistency for U (1)

When $\lambda^2 \in \mathbb{Z}$, the WZW model on U (1) has:

- A discrete spectrum.
- Completely reducible modules.
- A modular-invariant partition function

$$Z_\lambda[q] = \sum_{j=0}^{\lambda^2} \left| \chi_{W_{[j/\lambda]}}(q) \right|^2.$$

- A discrete Verlinde formula for extended algebra modules.

The algebraic picture then is that compactifying the target space leads to an extension of the symmetries by invertible elements of the fusion ring.

Example: $GL(1|1)$

We'd like to think now about the simplest examples of WZW models on Lie supergroups. Here, the algebra is under control but the geometry is poorly understood (by physicists). Nevertheless, we'll see that a parallel geometric formulation is expected.

Why do we want a geometric formulation?

- To get the spectrum in more general examples.
- To understand stringy motivations and applications to AdS/CFT.
- To extend to open strings and D-brane physics.
- To make contact with plenty of cool math — *eg.* Borel-Weil-Bott, Freed-Hopkins-Telemann.

Mostly, we just want to know everything...

Algebraic Preliminaries

In the absence of geometric understanding, let's try to build up a WZW model using only algebra...

$\mathfrak{gl}(1|1)$ has defining representation

$$\underbrace{E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{even}}, \quad \underbrace{\psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{odd}}$$

and non-zero (anti)commutators

$$\left[N, \psi^+\right] = 2\psi^+, \quad \left[N, \psi^-\right] = -2\psi^-, \quad \left\{\psi^+, \psi^-\right\} = E.$$

Verma modules $V_{n,e}$ are labelled by eigenvalues of N and E. They are irreducible if $e \neq 0$ (typical) and reducible if $e = 0$ (atypical).

Representation Rings

Sadly, the representation ring of $\mathfrak{gl}(1|1)$ generated by irreducibles contains non-semisimple modules:

$$V_{n,e} \otimes V_{n',-e} = P_{n+n'} \qquad (e \neq 0).$$

Here, P_n denotes an indecomposable with Loewy diagram

$$\begin{array}{ccc} & A_n & \\ A_{n+1} & & A_{n-1} \\ & A_n & \end{array}$$

where A_n denotes the irreducible quotient of $V_{n+1/2,0}$.

P_n is the projective cover of A_n (and $V_{n-1/2}$).

The Affine Kac-Moody Superalgebra

The non-vanishing relations of the affine modes are

$$\left[N_r, E_s\right] = rk\delta_{r+s,0}, \qquad \left[N_r, \psi_s^{\pm}\right] = \pm\psi_{r+s}^{\pm},$$
$$\left\{\psi_r^+, \psi_s^-\right\} = E_{r+s} + rk\delta_{r+s,0}$$

and we again have Virasoro modes

$$L_n = \frac{1}{k}\sum_{r\in\mathbb{Z}} : N_r E_{n-r} - \psi_r^+ \psi_{n-r}^- : - \frac{1}{2k}E_n + \frac{1}{2k^2}\sum_{r\in\mathbb{Z}} : E_r E_{n-r} :$$

with $c = 0$.

Now, the Verma modules $V_{n,e}$ are irreducible (typical) for $e/k \notin \mathbb{Z}$ and are reducible (atypical) for $e/k \in \mathbb{Z}$. Atypical irreducible quotients are denoted by $A_{n,e}$ and their projective covers by $P_{n,e}$.

Supercharacters

Supercharacters are obtained by taking supertraces:

$$\mathrm{str}_V M = \mathrm{tr}_{V_{\mathrm{even}}} M - \mathrm{tr}_{V_{\mathrm{odd}}} M, \qquad V = V_{\mathrm{even}} \oplus V_{\mathrm{odd}},$$
$$\chi_V(x; y; z; q) = \mathrm{str}_V\, x^k y^{E_0} z^{N_0} q^{L_0 - c/24}.$$

Typical irreducibles (and Verma modules) have supercharacter

$$\chi_{V_{n,e}}(x; y; z; q) = \mathrm{i} x^k y^e z^n q^{ne/k + e^2/2k^2} \frac{\vartheta_1(z; q)}{\eta(q)^3},$$

where $\vartheta_1(z; q)$ is a Jacobi theta function. Atypical characters are obtained from a BGG-resolution, *eg.* :

$$\cdots \longrightarrow V_{n-5/2,0} \longrightarrow V_{n-3/2,0} \longrightarrow V_{n-1/2,0} \longrightarrow A_{n,0} \longrightarrow 0$$
$$\Rightarrow \qquad \chi_{A_{n,0}} = \sum_{j=0}^{\infty} (-1)^j \chi_{V_{n-j-1/2,0}}.$$

Consistency Conditions 1

A simple suggestion for the modular invariant is

$$Z[x;y;z;q] = \iint_{\mathbb{R}^2} \left|\chi_{V_{n,e}}(x;y;z;q)\right|^2 \frac{\mathrm{d}n\,\mathrm{d}e}{k}.$$

The atypical irreducibles contribute by decomposing the characters of the atypical Verma modules.

This will be modular-invariant if the S-matrix is unitary:

$$x = \mathrm{e}^{2\pi\mathrm{i}t}, \quad y = \mathrm{e}^{2\pi\mathrm{i}\nu}, \quad z = \mathrm{e}^{2\pi\mathrm{i}\mu}, \quad q = \mathrm{e}^{2\pi\mathrm{i}\tau},$$

$$\chi_{V_{n,e}}\left(t - \frac{\mu\nu}{\tau}; \frac{\nu}{\tau}; \frac{\mu}{\tau}; -\frac{1}{\tau}\right) = \iint_{\mathbb{R}^2} \mathsf{S}_{(n,e)(n',e')} \chi_{V_{n',e'}}(t;\nu;\mu;\tau) \frac{\mathrm{d}n'\,\mathrm{d}e'}{k},$$

$$\iint_{\mathbb{R}^2} \mathsf{S}_{(n,e)(n',e')} \mathsf{S}^*_{(n'',e'')(n',e')} \frac{\mathrm{d}n'\,\mathrm{d}e'}{k} = \delta\left(n''=n\right)\delta\left(e''=e\right).$$

Consistency Conditions 2

Creutzig and DR found the S-matrix. Recall that

$$\chi_{V_{n,e}}(x; y; z; q) = \mathrm{i} x^k y^e z^n q^{ne/k + e^2/2k^2} \frac{\vartheta_1(z; q)}{\eta(q)^3},$$

$$\vartheta_1\left(\frac{\mu}{\tau}; -\frac{1}{\tau}\right) = \sqrt{-\mathrm{i}\tau}\, \mathrm{e}^{\mathrm{i}\pi\mu^2/\tau} \vartheta_1(\mu; \tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-\mathrm{i}\tau}\, \eta(\tau).$$

Then, the S-matrix is

$$\mathsf{S}_{(n,e)(n',e')} = -\mathrm{i}\omega \mathrm{e}^{-2\pi\mathrm{i}\left(ne'/k + n'e/k + ee'/k^2\right)} \qquad (|\omega| = 1)$$

which is indeed unitary.

This suggests that the quantum state space for GL $(1|1)$ is

$$\mathcal{H} \sim \iint_{\mathbb{R}^2} V_{n,e} \otimes V_{n,e} \frac{\mathrm{d}n\, \mathrm{d}e}{k}.$$

Consistency Conditions 3

The other consistency check is the Verlinde formula. Recall that

$$\chi_{A_{0,0}} = \sum_{j=0}^{\infty} (-1)^j \chi_{V_{-j-1/2,0}} \quad \Rightarrow \quad \mathsf{S}_{0(n,e)} = \frac{\omega}{2\sin(\pi e/k)}.$$

The Verlinde formula therefore reads

$$\mathbf{N}_{(n_1,e_1)(n_2,e_2)}{}^{(n_3,e_3)} = \iint_{\mathbb{R}^2} \frac{\mathsf{S}_{(n_1,e_1)(n,e)} \mathsf{S}_{(n_2,e_2)(n,e)} \mathsf{S}^*_{(n,e)(n_3,e_3)}}{\mathsf{S}_{0(n,e)}} \frac{\mathrm{d}n\,\mathrm{d}e}{k}$$

$$= \delta\Big(\frac{e_1+e_2-e_3}{k}\Big)\Big[\delta\big(n_1+n_2-n_3+1/2\big) - \delta\big(n_1+n_2-n_3-1/2\big)\Big]$$

$$\Rightarrow \qquad [V_{n_1,e_1}] \times [V_{n_2,e_2}] = [V_{n_1+n_2+1/2,e_1+e_2}] \ominus [V_{n_1+n_2-1/2,e_1+e_2}]$$

(in the Grothendieck ring of fusion). The negative integer multiplicity here is consistent with supercharacter parity!

Compactifying GL (1|1)

The WZW model on GL (1|1) has:

- A continuous spectrum.
- Indecomposable modules.
- A modular-invariant partition function.
- A continuum Verlinde formula.

This is identical to GL (1), except for the appearance of indecomposables.

Are there other Lie supergroups with the same Lie superalgebra? Yes. In particular, we can (partially) compactify.

We expect (partial) quantisation of the highest weights. Algebraically, we can ask for simple currents (invertible elements of the fusion ring) in order to construct extended algebras.

Algebraic Compactification

Verlinde doesn't give us the fusion ring, just its Grothendieck quotient. But, this suffices to determine invertible elements: All atypical irreducibles $A_{n,e}$ are simple currents.

Extending by appropriate atypicals gives a doubly-infinite family of VOAs including:

- $\beta\gamma$ ghosts.
- $\widehat{\mathfrak{sl}}(2|1)$ at levels $-\frac{1}{2}$ and 1.
- The Bershadsky-Polyakov algebra $W_3^{(2)}$ at levels 0 and $-\frac{5}{3}$.
- The $N = 2$ superconformal algebra of central charge ± 1.
- The Feigin-Semikhatov algebras $W_n^{(2)}$ of various levels.

For these cases, the untwisted extended modules form a discrete set in one "direction" and a continuum in the other.

Trouble in Paradise

For these compactified theories, one has:

- A (partially) discrete spectrum.
- Reducible but indecomposable modules, but confined to the atypical sector.
- Extended characters which are *mock* modular forms.
- Modular-invariant partition functions.

The Verlinde formula, however, fails miserably! This is a well-known problem in rational logarithmic CFT which indicates that our understanding of this consistency requirement needs refining.

Examples like these $GL(1|1)$ compactifications will hopefully suggest said refinements.

Conclusions

The algebraic features of the $\mathrm{GL}(1|1)$ WZW model have much in common with those of the $\mathrm{GL}(1)$ model:

- Continuous spectrum.
- Characters are modular forms.
- S-transforms are Fourier transforms.
- S-matrix is unitary (partition function is modular invariant).
- Continuum Verlinde formula gives (Grothendieck) fusion.
- Many simple currents giving extended algebras with modular-invariant partition functions.

This strongly suggests that one should have a geometric interpretation in terms of compactification/orbifolding/*etc*...

Speculation

Advantages of a geometric understanding would include:

- Analysis of quantisation constraints.
- Derivation of the spectrum.
- More direct contact with stringy applications (*eg*. AdS/CFT).
- Direct contact with applications to algebraic geometry, *etc*...

Some questions I have:

- Would indecomposable modules be distinguishable in geometric quantisation?
- Can geometric quantisation resolve the indecomposable structure in the atypical sectors?
- Can we study D-branes in super-WZW models and study brane charges, twisted K-theory, *etc*... ?