A higher chromatic analogue of the J-homomorphism

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Stable homotopy groups

Recall: For a topological space X, the stable homotopy groups of X are

$$\pi_k^S(X) := \lim_{n\to\infty} \pi_{n+k}\Sigma^n X = \pi_k\left(\lim_{n\to\infty} \Omega^n\Sigma^n X\right) =: \pi_k(QX)$$

where $\Omega^n Y = Map_*(S^n, Y)$.

Completely unattainable goal: Compute $\pi_k^S(S^0)$, $\forall k$.

Chromatic homotopy theory: Separates $\pi_k^S(S^0)$ into "chromatic layers," and then seeks to understand these layers. Classically, the Adams conjecture on the image of J is rephrased as a question about the *first* chromatic layer.

The J-homomorphism

Define $J_n : O(n) \to \Omega^n S^n$ as the map

$$(M : \mathbb{R}^n \to \mathbb{R}^n) \mapsto (J_n(M) = M \cup \{\infty\} : S^n \to S^n).$$

In the limit over n, we get $J : O \to QS^0$.

Theorem (Bott periodicity)

k mod 8	0	1	2	3	4	5	6	7
$\pi_k O$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Theorem (Adams, Quillen)

$\pi_* J$ is an injection if $* = 0, 1$ mod 8. Further, in dimension $4n - 1$, im$(\pi_* J)$ is $\mathbb{Z}/m$ where m is the denominator of $B_{2n}/4n$.

Equivalently ($p > 2$): im$(\pi_* J)_p = \mathbb{Z}/p^{k+1}$ if $* + 1 = 2(p-1)p^k m$, where m is coprime to p.

Idea of the proof

Let $K = K\mathbb{Z}_p$ be p-adic K-theory, and $K(1) = K/p$ be mod p K-theory.

Then, for each $k \in \mathbb{Z}_p^\times$, there is a natural transformation (*Adams operation*) $\psi^k : K \to K$, uniquely characterised by

1. $\psi^k : K^*(X) \to K^*(X)$ is a ring homomorphism.
2. $\psi^k \circ \psi^l = \psi^{kl}$.
3. $\psi^k(L) = L^{\otimes k}$ when $L \to X$ is a line bundle.

Theorem

There is a ring isomorphism $K(1)_*K \cong C(\mathbb{Z}_p^\times, \mathbb{F}_p)$ which carries ψ^k to the operation $\psi^k(f)(x) = f(kx)$.

Here $C(\mathbb{Z}_p^\times, \mathbb{F}_p)$ is the ring of continuous functions $f : \mathbb{Z}_p^\times \to \mathbb{F}_p$. We recall that $\mathbb{Z}_p^\times$ is topologically cyclic, with topological generator $g := \zeta(1+p)$, where $\zeta \in \mathbb{Z}_p^\times$ is a primitive $p-1^{\text{st}}$ root of unity.

Note: The map $\psi^g - 1 : K \to K$ is surjective in $K(1)_*$ with kernel consisting of those functions $f : \mathbb{Z}_p^\times \to \mathbb{F}_p$ with $f(gx) = f(x)$; i.e., the constant functions. This is rank 1 over $\mathbb{F}_p$.

Thus the homotopy fibre of $\psi^g - 1$ has $K(1)_*$ which is rank one, the same as that of a sphere.

Corollary

There is a fibre sequence $L_{K(1)}S^0 \longrightarrow K \xrightarrow{\psi^g - 1} K$

One checks that $\psi^g - 1 : \pi_{2j}K \to \pi_{2j}K$ is multiplication by p^{k+1} in $\mathbb{Z}_p$ when $j = (p-1)p^k m$. The long exact sequence in homotopy in dimension $2k$ is then:

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}_p \xrightarrow{p^{k+1}} \mathbb{Z}_p \longrightarrow \mathbb{Z}/p^{k+1} \longrightarrow 0 \longrightarrow \cdots$$

Moral: The image of J is the homotopy of the $K(1)$-local sphere (in positive dimensions).

Interlude on localisation

For a cohomology theory E, a spectrum X is *E-acyclic* if $E_*X = 0$. Another spectrum Y is *E-local* if for all E-acyclics X, $[X, Y] = 0$.

Examples

- $\widetilde{H}_*(\mathbb{R}P^\infty, \mathbb{Q}) = 0$, so $\Sigma^\infty \mathbb{R}P^\infty$ is $H\mathbb{Q}$-acyclic.
- E is definitionally E-local.

Write $E - Loc$ for the subcategory of *Spectra* consisting of E-local spectra.

Theorem (Bousfield)

There exists a functor L_E : Spectra $\to E - Loc$ which is the identity on $E - Loc$.

The chromatic program: understand $\pi_*^S(S^0)$ via $\pi_*(L_{K(n)}S^0)$, where $K(n)$ is height n *Morava K-theory*.

A generalisation of the J-homomorphism

Theorem (Morava, Hopkins-Miller)

There is a spectrum E_n and a group G_n acting on E_n with

$$L_{K(n)}S^0 \simeq E_n^{hG_n}.$$

- When $n = 1$, $E_1 = K$ and $G_1 = \mathbb{Z}_p^\times$, giving the above.
- The homotopy of E_n is

$$\pi_* E_n \cong W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}].$$

  This is a complete local ring with residue field

$$\pi_* K(n) = \mathbb{F}_{p^n}[u^{\pm 1}].$$

- When $n > 1$, G_n is much more complicated than $\mathbb{Z}_p^\times$: the units in a maximal order in a rank n^2 p-adic division algebra.

Motivation

Our goal: Reproduce some version of the K-theory calculation (i.e., using $\mathbb{Z}_p^\times$, not G_n) at higher chromatic levels. Our method is more naive, and based around the following:

Theorem (Snaith)

There is a map $\beta : S^2 \to \Sigma^\infty \mathbb{C}P^\infty_+$ such that

$$K \simeq \Sigma^\infty \mathbb{C}P^\infty_+[\beta^{-1}].$$

Idea: $\Sigma^\infty \mathbb{C}P^\infty_+$ is the "free cohomology theory generated by line bundles." This "generates" K by the splitting principle. Further, the invertibility of β (the *Bott class*) forces (complex) Bott periodicity.

A higher chromatic analogue should have:

- An analogue of the Bott map and Snaith's theorem.
- $\mathbb{Z}_p^\times$ worth of "Adams operations."
- An interpretation in terms of continuous functions on $\mathbb{Z}_p^\times$.

The Picard group

For a symmetric monoidal category $(C, \otimes)$ with unit S, one defines an abelian group

$$\mathrm{Pic}(C) := \{X \in C \mid \exists Y \text{ such that } X \otimes Y \cong S\}/\cong$$

Examples

1. $\mathrm{Pic}(Vect/k) = \{k\} \cong 0$.
2. $\mathrm{Pic}(Vect(X)) \cong Pic(X)$ (line bundles).
3. $\mathrm{Pic}(Spectra) = \{S^n\} \cong \mathbb{Z}$.
4. $\mathrm{Pic}_n := \mathrm{Pic}(K(n) - Loc)$ is very interesting. For instance, $\mathrm{Pic}_1 = \mathbb{Z}_p \oplus \mathbb{Z}/(2p-2)$ if $p > 2$ (Hopkins-Mahowald-Sadofsky).

An example is the *Gross-Hopkins* or *Brown-Comenetz* dual of S^0. The functor $I(X) = \mathrm{Hom}(\pi_*(X), \mathbb{Q}/\mathbb{Z})$ satisfies Brown-representability, and so is represented by a spectrum I; $I(X) = [X, I]$. It turns out that $L_{K(n)} I \in \mathrm{Pic}_n$.

Theorem (W.)

There exists $G \in \mathrm{Pic}_n$ and $\rho : G \to \Sigma^\infty L_{K(n)} K(\mathbb{Z}_p, n+1)_+ =: X_n$ with the following properties:

1. $K(n)_*(X_n[\rho^{-1}]) \cong C(\mathbb{Z}_p^\times, \mathbb{F}_p)$.
2. *The natural action of $k \in \mathbb{Z}_p^\times$ on $X_n[\rho^{-1}]$ yields, in $K(n)_*$, the formula $\psi^k f(x) = f(kx)$.*
3. *If $n = 1$, then $G = S^2$ and $\rho = \beta$.*
4. *If $p > \frac{n^2+n+2}{2}$, then $G = \Sigma^{2\frac{p^n-1}{p-1}+n-n^2} L_{K(n)} I$.*
5. $[G^j, X_n[\rho^{-1}]] \cong \mathbb{Z}_p$ *for every $j \in \mathbb{Z}$.*

Parts 1 and 2 give a fibre sequence $L_{K(n)} S^0 \longrightarrow X_n[\rho^{-1}] \xrightarrow{\psi^g - 1} X_n[\rho^{-1}]$, just as before. Property 5 then implies:

Corollary

$[G^j, L_{K(n)} S^0] = \mathbb{Z}/p^{k+1}$ *if $j + 1 = (p-1)p^k m$, where m is coprime to p.*

Questions

1. What sort of cohomology theory is $X_n[\rho^{-1}]$? Does it have anything to do with n-bundle gerbe modules?
2. Is there a concrete description of G? That is, can we understand the "Bott periodicity" that we've imposed upon $X_n[\rho^{-1}]$?
3. Can we conclude anything about $\pi_*(S^0)$, $K(n)$-locally?

Thanks

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