Norm-square localization for Hamiltonian \( LG \)-spaces

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Norm-square localization for Hamiltonian $G$-spaces

- $M$ Hamiltonian $G$-space, proper moment map $\mu : M \to \mathfrak{g}^*$
- Norm $|\cdot|$ on $\mathfrak{g}^*$

**Definition**

$|\mu|^2 : M \to \mathbb{R}$ is the **norm-square of the moment map**.
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Some past work:

- Kirwan (1984): Morse theory with $|\mu|^2$, Kirwan surjectivity
- Witten (1992): certain integrals localize to $\text{Crit}(|\mu|^2)$
- Paradan (1999, 2000): detailed norm-square localization formula
- Woodward (2005), Harada-Karshon (2012): other approaches
Hamiltonian $LG$-spaces

- $G$ compact Lie group, Lie algebra $\mathfrak{g}$
- Ad-invariant inner product $\langle -, - \rangle$
- $LG = \text{Map}(S^1, G)$ loop group
- $L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$, $LG$ acts by gauge transformations:

$$g \cdot \xi = \text{Ad}_g \xi - dgg^{-1}.$$
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  $g \cdot \xi = \text{Ad}_g \xi - dg g^{-1}$.

Definition

A Hamiltonian $LG$-space $(\mathcal{M}, \omega_{\mathcal{M}}, \Psi)$ consists of a symplectic Banach manifold, equipped with an $LG$-action, and with a proper moment map $\Psi : \mathcal{M} \to L\mathfrak{g}^*$, equivariant for the gauge action of $LG$ on $L\mathfrak{g}^*$.

Similar to finite dimensions: convexity theorem, cross-sections, etc.
$L^2$ norm on $Lg^*$ by integration:

$$||\gamma||^2 = \frac{1}{2\pi} \int_0^{2\pi} |\gamma(\theta)|^2 d\theta.$$ 

$$||\Psi||^2 : M \rightarrow \mathbb{R}$$

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- $t_+$ positive Weyl chamber
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**Theorem (Kirwan, Bott-Tolman-Weitsman)**

*The critical set of $||\psi||^2$ is*

$$\text{Crit}(||\psi||^2) = \bigcup_{\beta \in \mathcal{B}} G \cdot (\mathcal{M}^\beta \cap \psi^{-1}(\beta)),$$

where $\mathcal{B} \subset t^*_+$ is a discrete subset.
Quasi-Hamiltonian $G$-spaces

- $\theta^L = g^{-1} dg$, $\theta^R = dgg^{-1}$ the left, resp. right Maurer-Cartan forms on $G$
- $\eta = \frac{1}{12} \langle [\theta^L, \theta^L], \theta^L \rangle$ Cartan 3-form
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Definition (Alekseev, Malkin, Meinrenken)

A quasi-Hamiltonian $G$-space $(M, \omega, \Phi)$ is a $G$-manifold $M$, equipped with a $G$-invariant 2-form $\omega$ together with an equivariant map $\Phi : M \to G$ satisfying

1. $d\omega = \Phi^* \eta$
2. $\iota_{\xi_M} \omega = -\frac{1}{2} \Phi^* \langle \theta^L + \theta^R, \xi \rangle$
3. $\text{ker}(\omega) \cap \text{ker}(d\Phi) = 0$.

Reduction: at conjugacy class $C \subset G$: $\Phi^{-1}(C)/G$, is symplectic.

Examples: conjugacy classes, moduli spaces of flat connections on Riemann surfaces, even-dimensional spheres.
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\[
\begin{array}{ccc}
M & \xrightarrow{\Psi} & Lg^* \\
\downarrow /L_0G & & \downarrow /L_0G \\
M & \xrightarrow{\Phi} & G
\end{array}
\]

$L_0G = \{ \gamma \in LG | \gamma(0) = \gamma(1) = e \}$
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\mathcal{M} & \xrightarrow{\Phi} & G \\
\end{array} \]

- $\omega_M$ not basic—must be modified to $\omega_M - \Psi^*\overline{\omega}$, which then descends to $\mathcal{M}$.
- Reduced spaces agree, e.g. $\Psi^{-1}(0)/G \simeq \Phi^{-1}(e)/G$. 
Possible degeneracy ⇒ need to modify formula for volume form.
Volume forms

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Adjoint action (assume \(G\) connected)

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Recall:

$$\text{Spin}(V) \subset \text{Cliff}(V), \quad \text{Cliff}(V) \cong \wedge V.$$
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Assume lift exists:

$$\tilde{\text{Ad}} : G \rightarrow \text{Spin}(\mathfrak{g}).$$

Recall: $\text{Spin}(V) \subset \text{Cliff}(V)$, $\text{Cliff}(V) \simeq \wedge V.$

Compose to get map $\psi : G \rightarrow \wedge \mathfrak{g}$, i.e. a left-invariant differential form on $G$. 
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Suppose $G$ simply connected. The top degree form

$$\Gamma = (e^\omega \Phi^* \psi)^{[\text{top}]}$$

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is a volume form on $M$.

Example

$G$ simple, simply connected. Fundamental alcove $\mathcal{A} \subset t_+$. Conjugacy class $\mathcal{C}_\mu$ containing $\exp(\mu)$, $\mu \in \mathcal{A}$.

$$\text{Vol}(\mathcal{C}_\mu) = \text{vol}(G/G_{\exp(\mu)}) \prod_{\alpha > 0, \langle \alpha, \mu \rangle \notin \mathbb{Z}} 2\sin(\pi \langle \alpha, \mu \rangle).$$
Duistermaat-Heckman distributions

**Definition**

The *Duistermaat-Heckman (DH) distribution* of a quasi-Hamiltonian space is the pushforward

$$\Phi_* |\Gamma| \in \mathcal{D}'(G)^G.$$
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Volume of reduced spaces: assume \( e \in G \) regular value \( \Rightarrow \)

\[ \text{vol}(\Phi^{-1}(e)/G) = \frac{d}{\text{vol}(G)} \frac{\Phi_*|\Gamma|}{d\text{vol}_G} \bigg|_e. \]

Remark

More general DH distributions twisted by \( \alpha \in H_G(M) \) encode cohomology pairings on quotients.
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Remark

More general DH distributions twisted by \( \alpha \in H_G(M) \) encode cohomology pairings on quotients.

Next define related distribution on \( T \subset G \).
The map $R_G$

- $W = N(T)/T$ Weyl group
- $\rho$ half-sum pos. roots, $n_+$ number of pos. roots
- $\chi_\lambda$ irreducible rep. corresponding to dominant weight $\lambda$

**Definition**

There is an isomorphism

$$R_G : \mathcal{D}'(G)^G \sim \to \mathcal{D}'(T)^{W-\text{anti}}.$$ 

determined by the equation

$$i^{-n_+} \langle n, \overline{\chi_\lambda} \rangle = \text{vol}_{G/T} \langle R_G(n), t^{\lambda+\rho} \rangle.$$
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**Example**

Let $f \in C^\infty(G)^G$, and $J(t) = \sum_{w \in W} (-1)^{|w|} t^\rho$.

$$R_G(f \ d\text{vol}_G) = J \cdot (f|_T) \ d\text{vol}_T.$$
Example

Let \( \mu \in \text{int}(\mathcal{A}) \) and 
\[
\iota : C_\mu \hookrightarrow G.
\]
\( \iota_*|\Gamma| \in \mathcal{D}'(G)^G \) is a delta distribution on \( C_\mu \subset G \) with total weight \( \text{Vol}(C_\mu) \). Then:
\[
R_G(\iota_*|\Gamma|) = \frac{1}{|W|} \sum_{w \in W} (-1)^{|w|} \delta_w \exp \mu.
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Example

Let $\mu \in \text{int}(\mathcal{A})$ and

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$\iota_*|\Gamma| \in \mathcal{D}'(G)^G$ is a delta distribution on $C_\mu \subset G$ with total weight $\text{Vol}(C_\mu)$. Then:

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Example

$e \in G$ identity element

$$R_G(\delta_e^G) = \left( \prod_{\alpha < 0} \partial_\alpha \right) \delta_e^T.$$
A DH distribution for $\Phi^{-1}(T)$

**Definition**

Recall $\Phi_*|\Gamma| \in \mathcal{D}'(G)^G$ is the DH distribution of $M$. We define

$$m = R_G(\Phi_*|\Gamma|) \in \mathcal{D}'(T)^{W-\text{anti}}.$$ 

$m$ is the distribution we will discuss for the rest of the talk.
A DH distribution for $\Phi^{-1}(T)$

### Definition
Recall $\Phi_\ast |\Gamma| \in \mathcal{D}'(G)^G$ is the DH distribution of $M$. We define

$$m = R_G(\Phi_\ast |\Gamma|) \in \mathcal{D}'(T)^{W-\text{anti}}.$$ 

$m$ is the distribution we will discuss for the rest of the talk.

### Basic properties:
- $\Phi$ transverse to $T \Rightarrow m$ is DH distribution of $\Phi^{-1}(T) \subset M$.
- Gives $\text{vol}(\Phi^{-1}(C_\mu)/G)$ directly for $\mu \in \text{int}(\mathcal{A})$.
- If $\Phi$ has regular values, it is piecewise-polynomial.
Examples of $m \in \mathcal{D}'(T)^{W\text{-anti}}$

**Example**

\[ DSU(2) = SU(2) \times SU(2) \ominus SU(2), \quad \Phi(a, b) = aba^{-1}b^{-1} \]

$|\Gamma|$ = Haar measure  \quad \quad \quad \quad \quad \quad m = R_G(\Phi_*|\Gamma|)$ is:
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\]

Example

\[S^4 \odot SU(2) \quad m = R_G(\Phi_\star |\Gamma|) \text{ is:}
\]
Example of $m \in \mathcal{D}'(T)^{W-\text{anti}}$

**Example**

Multiplicity-free, quasi-Hamiltonian $SU(3)$-space (Chris Woodward).
Recall: \[ \text{Crit}(\|\Psi\|^2) = \bigcup_{\beta \in \mathcal{B}} G \cdot (\mathcal{M}^\beta \cap \Psi^{-1}(\beta)) \]

**Theorem**

*There is a norm-square localization formula for \( m \):*

\[
m = \sum_{\beta \in W \cdot B} m_\beta.
\]

- \( m_\beta \) piecewise-polynomial on cones with apex at \( \beta \).
- \( \beta \neq 0 \Rightarrow \text{support } m_\beta \text{ contained in half-space } \beta \geq \|\beta\|^2. \)
- Expression for \( m_\beta \) as integral over submanifold near \( \mathcal{M}^\beta \cap \Psi^{-1}(\beta) \), involving local geometric data.
\[ \phi : Y \to U \text{ cross-section, } \beta \in U \subset Lg^*, \text{ project } \phi^t := \text{pr}^t \circ \phi. \]

Minimal coupling $\to \text{Tot}(\nu(Y^\beta, Y^\beta))$ becomes Hamiltonian $T$-space (polarized weights).

For $m^\beta$: take germ of (twisted) DH measure for $\text{Tot}(\nu(Y^\beta, Y^\beta))$ near $\beta$ and extend.

Explicit formula similar to Paradan (2000). Convolutions of polynomial distributions on walls, and Heaviside distributions $H^\alpha, \langle \alpha, \beta \rangle > 0$. 

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July 2015 16 / 34
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Norm-square localization for $S^4$
Chris Woodward’s example
First three contributions
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Outline of method

Strategy modelled on that of Szenes-Vergne ("[Q,R]=0 and Kostant multiplicity functions", 2010).
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- Write \(m\) as sum of simpler distributions (abelian localization).

  Do norm-square localization for each of these simpler distributions. Because they are so simple, this can be done with combinatorics! (But the result is a mess...)

  Use geometry + abelian localization (again) to re-assemble/re-interpret terms.

  For explicit expressions (similar to Paradan (2000)) involving an integral near the critical set, some additional argument involved (more technical, not for today).
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Example: $S^4$
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Add central contributions:

Next two contributions (along positive $\mathbb{R}$ axis):
Chris Woodward’s example

3 Fixed-point contributions:

- Pullbacks of
- Central contribution from
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Central contribution from each is a non-trivial linear function. But the sum is zero.
$\rightarrow \text{Identify } t \simeq t^*.$

**Theorem (Alekseev, Meinrenken, Woodward)**

Let $\xi \in \Lambda^\ast$ (weight lattice). We have the following abelian localization formula for the Fourier coefficients of $m$:

$$\langle m, t^\xi \rangle = \prod_{\alpha > 0} 2\pi i \langle \alpha, \xi \rangle \sum_{\nu \in \Lambda^\ast} \int_{F \subset M^\xi} e^{2\pi i \omega \Phi^\xi} \frac{e^{-2\pi i \langle \mu, \xi \rangle}}{\text{Eul}(\nu_F, \xi)}.$$

Want to interchange the two summations.
Abelian localization formula for $m$

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**Fourier inversion:**

\[
m(\mu) = \prod_{\alpha < 0} \partial_{\alpha} \sum_{\xi \in \Lambda^*} \sum_{F \subset M^\xi} \int_F \frac{e^{2\pi i \omega \Phi^\xi}}{\text{Eul}(\nu_F, \xi)} e^{-2\pi i \langle \mu, \xi \rangle}. \]

Want to interchange the two summations.
Abelian localization formula for $m$

Take $F$, closure of a $T$-orbit type.

- $t_F \subset t$ infinitesimal stabilizer of $F$
- $\alpha_i, i = 1, \ldots, N$ list of weights on the normal bundle $\nu_F$
- $(\Lambda^* \cap t_F) \setminus \cup \{\alpha_i = 0\}$ subset of $\Lambda^*$ where $\int_F$ appears

Then:

$$m = \prod_{\alpha < 0} \frac{\partial \alpha}{\sum_{F} m_F(\mu)}$$

where $$m_F(\mu) = \sum_{\xi' \in \Lambda^* \cap t_F} \int_F e^{-\frac{1}{2\pi i} \langle \mu, \xi \rangle} Eul(\nu_F, \xi) e^{2\pi i \omega \Phi \xi}.$$
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$\Rightarrow m_F$ is sum of (shifted) multiple Bernoulli series.
Multiple Bernoulli series

- $V$ vector space, $\Gamma \subset V$ lattice, dual $\Gamma^* \subset V^*$
- $\alpha$ a list of elements of $\Gamma^*$

**Definition (Szenes (1998), Brion-Vergne (1999),...)**

The **multiple Bernoulli series** associated to the data $V, \Gamma, \alpha$ is:

$$B_{\alpha, \Gamma}(\mu) = \sum'_{\xi \in \Gamma} \frac{e^{2\pi i \langle \mu, \xi \rangle}}{\prod_k 2\pi i \langle \alpha_k, \xi \rangle}.$$
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**Examples**

- $B_{\emptyset, \mathbb{Z}}(x) = \sum_{n \in \mathbb{Z}} e^{2\pi inx} = \sum_{n \in \mathbb{Z}} \delta(x - n)$
- $B_{1, \mathbb{Z}}(x) = \sum_{n \neq 0} \frac{e^{2\pi inx}}{2\pi in} = \frac{1}{2} - x + \lfloor x \rfloor$
Choose generic $\gamma \in V^*$, and inner product.

**Theorem (Boysal-Vergne)**

There is a decomposition:

$$B_{\alpha, \Gamma} = \sum_{\Delta \in \mathcal{A}} B_{\alpha, \Gamma, \Delta},$$

where

- $\mathcal{A}$ an infinite collection of affine subspaces $\Delta \subset V^*$,
- $B_{\alpha, \Gamma, \Delta}$ is a convolution of a polynomial distribution on $\Delta$ with Heaviside distributions in transverse directions,
- for $\Delta \neq V^*$, $\gamma \notin \text{support}(B_{\alpha, \Gamma, \Delta})$. 

Decomposition of $m_F$

Recall:

$$m = \prod_{\alpha < 0} \partial_\alpha \sum_F m_F,$$

where

$$m_F(\mu) = \sum'_{\xi \in \Lambda^* \cap t_F} \int_F \frac{e^{-2\pi i \langle \mu, \xi \rangle}}{\Eul(\nu_F, \xi)} e^{2\pi i \omega \Phi_\xi}.$$
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- Apply Boysal-Vergne type decomposition.
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- Apply Boysal-Vergne type decomposition.
- Group terms according to affine subspaces $\Delta$. 
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$$m_F(\mu) = \sum_{\xi \in \Lambda^* \cap t_F} \int_F \frac{e^{-2\pi i \langle \mu, \xi \rangle}}{\text{Eul}(\nu_F, \xi)} e^{2\pi i \omega \Phi_\xi}.$$

- Apply Boysal-Vergne type decomposition.
- Group terms according to affine subspaces $\Delta$.
- Subalgebra $\Delta^\perp =: t_\Delta \subset t$. 
Decomposition of $m_F$

Recall:

$$m = \prod_{\alpha < 0} \partial_{\alpha} \sum_{F} m_{F},$$

where

$$m_{F}(\mu) = \sum'_{\xi \in \Lambda^* \cap t_{F}} \int_{F} \frac{e^{-2\pi i \langle \mu, \xi \rangle}}{\text{Eul}(\nu_{F}, \xi)} e^{2\pi i \omega} \Phi_{\xi}.$$

- Apply Boysal-Vergne type decomposition.
- Group terms according to affine subspaces $\Delta$.
- Subalgebra $\Delta^\perp =: t_{\Delta} \subset t$.
- Further grouping according to lattice of $T$ orbit-types.

Then:

$$m = \sum_{\Delta} \sum_{C \subset M^{t\Delta}} m_{\Delta, C}.$$
- $C \subset M^\Delta$ is a \emph{quasi-Hamiltonian} space.
- $\text{Tot}(\nu(C, M))$ is a “hybrid” between Hamiltonian and quasi-Hamiltonian.
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- $\text{Tot}(\nu(C, M))$ is a “hybrid” between Hamiltonian and quasi-Hamiltonian.
- Let $\beta \in t_{\Delta}$ be orthogonal projection of 0 onto $\Delta$. 

\[ \beta \] 

\[ 0 \to t_{\Delta} \to t_{\Delta} = t \]
• $C \subset M^{t_{\Delta}}$ is a quasi-Hamiltonian space.
• $\text{Tot}(\nu(C, M))$ is a “hybrid” between Hamiltonian and quasi-Hamiltonian.
• Let $\beta \in t_{\Delta}$ be orthogonal projection of 0 onto $\Delta$.

$\Rightarrow$ Term $m_{\Delta, C}$ corresponding to $C \subset M^{t_{\Delta}}$ is the extension of the germ (at $\beta$) of a DH distribution for $\text{Tot}(\nu(C, M))$! (abelian localization on $\text{Tot}(\nu(C, M))$)
Reinterpretation of summands

\[ \Delta = t \]

\[ t_\Delta = t \]

- \( \Rightarrow \) contribution vanishes unless \( \Phi^{-1}(\exp(\beta)) \cap M^{t_\Delta} \neq \emptyset \). Since

\[ \Phi^{-1}(\exp(\beta)) \cap M^{t_\Delta} \simeq \Psi^{-1}(\beta) \cap M^{t_\Delta}, \]

- \( \Rightarrow \) non-zero contributions indexed by components of critical set of \( ||\Psi||^2 \).

- Further argument leads to explicit formulas involving integrals in cross-sections near critical set.