The Comparison of two constructions of the Refined Analytic Torsion on Manifolds with Boundary

Rung-Tzung Huang
j. w. w
Yoonweon Lee(Inha University, Korea)

Department of Mathematics
National Central University, Taiwan

IGA/AMSI workshop on geometric quantization
The University of Adelaide
7/27-31, 2015
Outline

1. Analytic torsion
2. Refined analytic torsion
3. Comparison theorem for refined analytic torsions
- $(M, g^M)$ a closed Riemannian manifold, $\dim(M) = m$.
- $(E, \nabla, h^E)$ a flat complex vector bundle over $M$, i.e. $\nabla^2 = 0$.
- In general $dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla' v)$, $\forall u, v \in C^\infty(M, E)$.
- $\nabla'$ is called the dual connection on $E$.
- If $\nabla$ Hermitian (i.e. $h^E$ flat), then $\nabla' = \nabla$.

**Fact**

- $(E, \nabla)$ is flat $\iff \exists \rho : \pi_1(M) \to GL(n, \mathbb{C})$ s.t. $E = \tilde{M} \times \rho \mathbb{C}^n$, where $\tilde{M}$ is the universal covering of $M$.
- $(E, \nabla)$ is flat and $h^E$ is flat $\iff \exists \rho : \pi_1(M) \to U(n)$ s.t. $E = \tilde{M} \times \rho \mathbb{C}^n$. 
Flat bundle

- $(M, g^M)$ a closed Riemannian manifold, $\dim(M) = m$.
- $(E, \nabla, h^E)$ a flat complex vector bundle over $M$, i.e. $\nabla^2 = 0$.
- In general $dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla' v), \forall u, v \in C^\infty(M, E)$.
- $\nabla'$ is called the dual connection on $E$.
- If $\nabla$ Hermitian (i.e. $h^E$ flat), then $\nabla' = \nabla$.

**Fact**

- $(E, \nabla)$ is flat $\iff \exists \rho : \pi_1(M) \to GL(n, \mathbb{C})$ s.t. $E = \tilde{M} \times_{\rho} \mathbb{C}^n$, where $\tilde{M}$ is the universal covering of $M$.
- $(E, \nabla)$ is flat and $h^E$ is flat $\iff \exists \rho : \pi_1(M) \to U(n)$ s.t. $E = \tilde{M} \times_{\rho} \mathbb{C}^n$. 
Flat bundle

- \((M, g^M)\) a closed Riemannian manifold, \(\dim(M) = m\).
- \((E, \nabla, h^E)\) a flat complex vector bundle over \(M\), i.e. \(\nabla^2 = 0\).
- In general \(dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla' v), \forall u, v \in C^\infty(M, E)\).
- \(\nabla'\) is called the dual connection on \(E\).
- If \(\nabla\) Hermitian (i.e. \(h^E\) flat), then \(\nabla' = \nabla\).

**Fact**

- \((E, \nabla)\) is flat \(\iff\) \(\exists \rho : \pi_1(M) \to GL(n, \mathbb{C})\) s.t. \(E = \tilde{M} \times_{\rho} \mathbb{C}^n\), where \(\tilde{M}\) is the universal covering of \(M\).
- \((E, \nabla)\) is flat and \(h^E\) is flat \(\iff\) \(\exists \rho : \pi_1(M) \to U(n)\) s.t. \(E = \tilde{M} \times_{\rho} \mathbb{C}^n\).
Flat bundle

- \((M, g^M)\) a closed Riemannian manifold, \(\dim(M) = m\).
- \((E, \nabla, h^E)\) a flat complex vector bundle over \(M\), i.e. \(\nabla^2 = 0\).
- In general \(dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla' v), \forall u, v \in C^\infty(M, E)\).
- \(\nabla'\) is called the dual connection on \(E\).
- If \(\nabla\) Hermitian (i.e. \(h^E\) flat), then \(\nabla' = \nabla\).

Fact

- \((E, \nabla)\) is flat if and only if \(\exists \rho : \pi_1(M) \to GL(n, \mathbb{C})\) s.t. \(E = \tilde{M} \times_\rho \mathbb{C}^n\), where \(\tilde{M}\) is the universal covering of \(M\).
- \((E, \nabla)\) is flat and \(h^E\) is flat if and only if \(\exists \rho : \pi_1(M) \to U(n)\) s.t. \(E = \tilde{M} \times_\rho \mathbb{C}^n\).
Flat bundle

- \((M, g^M)\) a closed Riemannian manifold, \(\dim(M) = m\).
- \((E, \nabla, h^E)\) a flat complex vector bundle over \(M\), i.e. \(\nabla^2 = 0\).
- In general \(dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla' v), \forall u, v \in C^\infty(M, E)\).
- \(\nabla'\) is called the dual connection on \(E\).
- If \(\nabla\) Hermitian (i.e. \(h^E\) flat), then \(\nabla' = \nabla\).

Fact

- \((E, \nabla)\) is flat \(\iff\) \(\exists \rho : \pi_1(M) \to GL(n, \mathbb{C})\) s.t. \(E = \tilde{M} \times_\rho \mathbb{C}^n\), where \(\tilde{M}\) is the universal covering of \(M\).
- \((E, \nabla)\) is flat and \(h^E\) is flat \(\iff\) \(\exists \rho : \pi_1(M) \to U(n)\) s.t. \(E = \tilde{M} \times_\rho \mathbb{C}^n\).
Flat bundle

- \((M, g^M)\) a closed Riemannian manifold, \(\dim(M) = m\).
- \((E, \nabla, h^E)\) a flat complex vector bundle over \(M\), i.e. \(\nabla^2 = 0\).
- In general, \(dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla' v), \forall u, v \in C^\infty(M, E)\).
- \(\nabla'\) is called the dual connection on \(E\).
- If \(\nabla\) Hermitian (i.e. \(h^E\) flat), then \(\nabla' = \nabla\).

Fact

- \((E, \nabla)\) is flat \(\iff\) \(\exists \rho : \pi_1(M) \rightarrow GL(n, \mathbb{C})\) s.t. \(E = \tilde{M} \times_{\rho} \mathbb{C}^n\), where \(\tilde{M}\) is the universal covering of \(M\).
- \((E, \nabla)\) is flat and \(h^E\) is flat \(\iff\) \(\exists \rho : \pi_1(M) \rightarrow U(n)\) s.t. \(E = \tilde{M} \times_{\rho} \mathbb{C}^n\).


- \((M, g^M)\) a closed Riemannian manifold, \(\dim(M) = m\).

- \((E, \nabla, h^E)\) a flat complex vector bundle over \(M\), i.e. \(\nabla^2 = 0\).

- In general \(dh^E(u, v) = h^E(\nabla u, v) + h^E(u, \nabla' v)\), \(\forall u, v \in C^\infty(M, E)\).

- \(\nabla'\) is called the dual connection on \(E\).

- If \(\nabla\) Hermitian (i.e. \(h^E\) flat), then \(\nabla' = \nabla\).

**Fact**

- \((E, \nabla)\) is flat \(\iff\) \(\exists \rho : \pi_1(M) \to GL(n, \mathbb{C})\) s.t. \(E = \tilde{M} \times_\rho \mathbb{C}^n\), where \(\tilde{M}\) is the universal covering of \(M\).

- \((E, \nabla)\) is flat and \(h^E\) is flat \(\iff\) \(\exists \rho : \pi_1(M) \to U(n)\) s.t. \(E = \tilde{M} \times_\rho \mathbb{C}^n\).
**Hodge theory**

- **de Rham complex:**

\[
0 \to \Omega^0(M, E) \xrightarrow{\nabla} \Omega^1(M, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^m(M, E) \to 0
\]

- **de Rham theorem:**

\[
H^p(M, E) \cong H^p_{dR}(M, E) = \frac{\text{Ker}(\nabla|_{\Omega^p(M, E)})}{\text{Im}(\nabla|_{\Omega^{p-1}(M, E)})}
\]

- **Hodge Laplacian:**

\[
\Delta_p = \nabla \nabla^* + \nabla^* \nabla : \Omega^p(M, E) \to \Omega^p(M, E),
\]

where $\nabla^*$ is the adjoint of $\nabla$ w.r.t. $\langle \cdot, \cdot \rangle$ on $\Omega^\bullet(M, E)$ induced from $g^M$ and $h^E$.

- **Hodge theorem:**

$H^p(M, E) \cong H^p_{dR}(M; E) \cong \text{Ker} \Delta_p$
Hodge theory

- de Rham complex:

\[0 \to \Omega^0(M, E) \xrightarrow{\nabla} \Omega^1(M, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^m(M, E) \to 0\]

- de Rham theorem:

\[H^p(M, E) \cong H^p_{dR}(M, E) = \frac{\text{Ker}(\nabla|_{\Omega^p(M, E)})}{\text{Im}(\nabla|_{\Omega^{p-1}(M, E)})}\]

- Hodge Laplacian:

\[\Delta_p = \nabla \nabla^* + \nabla^* \nabla : \Omega^p(M, E) \to \Omega^p(M, E),\]

where \(\nabla^*\) is the adjoint of \(\nabla\) w.r.t. \(\langle \cdot, \cdot \rangle\) on \(\Omega^\bullet(M, E)\) induced from \(g^M\) and \(h^E\).

- Hodge theorem:

\[H^p(M, E) \cong H^p_{dR}(M; E) \cong \text{Ker} \Delta_p\]
Hodge theory

- de Rham complex:

\[ 0 \to \Omega^0(M, E) \xrightarrow{\nabla} \Omega^1(M, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^m(M, E) \to 0 \]

- de Rham theorem:

\[ H^p(M, E) \cong H^p_{dR}(M, E) = \frac{\text{Ker}(\nabla|_{\Omega^p(M,E)})}{\text{Im}(\nabla|_{\Omega^{p-1}(M,E)})} \]

- Hodge Laplacian:

\[ \Delta_p = \nabla\nabla^* + \nabla^*\nabla : \Omega^p(M, E) \to \Omega^p(M, E), \]

where \( \nabla^* \) is the adjoint of \( \nabla \) w.r.t. \( \langle \cdot, \cdot \rangle \) on \( \Omega^\bullet(M, E) \) induced from \( g^M \) and \( h^E \).

- Hodge theorem: \( H^p(M, E) \cong H^p_{dR}(M; E) \cong \text{Ker} \Delta_p \)

Rung-Tzung Huang (NCU)
Hodge theory

- de Rham complex:

\[ 0 \rightarrow \Omega^0(M, E) \xrightarrow{\nabla} \Omega^1(M, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^m(M, E) \rightarrow 0 \]

- de Rham theorem:

\[ H^p(M, E) \cong H^p_{dR}(M, E) = \frac{\text{Ker}(\nabla|_{\Omega^p(M,E)})}{\text{Im}(\nabla|_{\Omega^{p-1}(M,E)})} \]

- Hodge Laplacian:

\[ \Delta_p = \nabla \nabla^* + \nabla^* \nabla : \Omega^p(M, E) \rightarrow \Omega^p(M, E), \]

where \( \nabla^* \) is the adjoint of \( \nabla \) w.r.t. \( < \cdot, \cdot > \) on \( \Omega^\bullet(M, E) \) induced from \( g^M \) and \( h^E \).

- Hodge theorem:

\[ H^p(M, E) \cong H^p_{dR}(M; E) \cong \text{Ker} \Delta_p \]
For $s \in \mathbb{C}$, $\text{Re } s > m/2$, the $\zeta$-function

$$\zeta_{\Delta_p}(s) := \text{Tr}(\Delta_p)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}[\exp(-t\Delta_p) - \dim \text{Ker } \Delta_p] \, dt$$

converges. Moreover, it has a meromorphic continuation to $\mathbb{C}$. In particular, it is regular at $s = 0$.

Define $\zeta$-regularized determinant

$$\text{Det } \Delta_p := \exp(-\zeta'_{\Delta_p}(0)).$$

Formally,

$$\text{Det } \Delta_p = \prod_{\lambda_k > 0} \lambda_k$$
For \( s \in \mathbb{C} \), \( \text{Re} \, s > m/2 \), the \( \zeta \)-function

\[
\zeta_{\Delta_p}(s) := \text{Tr}(\Delta_p)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}[\exp(-t\Delta_p) - \text{dim Ker} \, \Delta_p] \, dt
\]

converges. Moreover, it has a meromorphic continuation to \( \mathbb{C} \). In particular, it is regular at \( s = 0 \).

Define \( \zeta \)-regularized determinant

\[
\text{Det} \, \Delta_p := \exp(-\zeta'_{\Delta_p}(0)).
\]

Formally,

\[
\text{Det} \, \Delta_p = \prod_{\lambda_k > 0} \lambda_k
\]
For $s \in \mathbb{C}$, $\text{Re} \ s > m/2$, the $\zeta$-function

$$
\zeta_{\Delta_p}(s) := \text{Tr}(\Delta_p)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}[\exp(-t\Delta_p) - \dim \text{Ker} \Delta_p] \, dt
$$

converges. Moreover, it has a meromorphic continuation to $\mathbb{C}$. In particular, it is regular at $s = 0$.

Define $\zeta$-regularized determinant

$$
\text{Det} \Delta_p := \exp(-\zeta'_{\Delta_p}(0)).
$$

Formally,

$$
\text{Det} \Delta_p = " \prod_{\lambda_k > 0} \lambda_k "
$$
For \( s \in \mathbb{C}, \text{Re} s > m/2 \), the \( \zeta \)-function

\[
\zeta_{\Delta_p}(s) := \text{Tr}(\Delta_p)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}[\exp(-t\Delta_p) - \dim \text{Ker} \Delta_p] dt
\]

converges. Moreover, it has a **meromorphic continuation** to \( \mathbb{C} \). In particular, it is **regular** at \( s = 0 \).

Define **\( \zeta \)-regularized determinant**

\[
\text{Det} \Delta_p := \exp(-\zeta'_{\Delta_p}(0)).
\]

Formally,

\[
\text{Det} \Delta_p = " \prod_{\lambda_k > 0} \lambda_k "
\]
\( V: \) n-dim. vector space, \( \det V := \wedge^n V \) complex line.

**Volume element:** \([v] = v_1 \wedge \cdots \wedge v_n \in \det V, \)
where \( \{v_i\} \) orthonormal basis for \( V. \)

**Determinant line of cohomology groups:**

\[
\det H^\bullet(M, E) = \bigotimes_p (\det H^p(M, E))^{-1} \pmod{p},
\]

For \([h_i] \in \det H^i(M, E),\]

\[
\rho(\nabla, g^M) = [h_0] \otimes [h_1]^{-1} \otimes \cdots \otimes [h_m] \in \det H^\bullet(M, E).
\]
Determinant line

- $V$: n-dim. vector space, $\det V := \wedge^n V$ complex line.

- **Volume element:** $[v] = v_1 \wedge \cdots \wedge v_n \in \det V$, where $\{v_i\}$ orthornormal basis for $V$.

- Determinant line of cohomology groups:

  $$\det H^\bullet(M, E) = \bigotimes_p (\det H^p(M, E))^{(-1)^p},$$

- For $[h_i] \in \det H^i(M, E)$,

  $$\rho(\nabla, g^M) = [h_0] \otimes [h_1]^{-1} \otimes \cdots \otimes [h_m]^{\pm} \in \det H^\bullet(M, E).$$
Determinant line

- $V$: n-dim. vector space, $\det V := \wedge^n V$ complex line.
- **Volume element:** $[\mathbf{v}] = v_1 \wedge \cdots \wedge v_n \in \det V$, where $\{v_i\}$ orthornormal basis for $V$.
- **Determinant line** of cohomology groups:

$$\det H^\bullet(M, E) = \otimes_p (\det H^p(M, E))^{(-1)^p},$$

- For $[h_i] \in \det H^i(M, E)$,

$$\rho(\nabla, g^M) = [h_0] \otimes [h_1]^{-1} \otimes \cdots \otimes [h_m]^\pm \in \det H^\bullet(M, E).$$
Determinant line

- $V$: n-dim. vector space, $\det V := \wedge^n V$ complex line.

- **Volume element:** $[v] = v_1 \wedge \cdots \wedge v_n \in \det V$, where \{v_i\} orthonormal basis for $V$.

- **Determinant line of cohomology groups:**

$$\det H^\bullet(M, E) = \bigotimes_p (\det H^p(M, E))^{(-1)^p},$$

- For $[h_i] \in \det H^i(M, E)$,

$$\rho(\nabla, g^M) = [h_0] \otimes [h_1]^{-1} \otimes \cdots \otimes [h_m]^{\pm} \in \det H^\bullet(M, E).$$
Ray-Singer analytic torsion

- **Scalar Ray-Singer torsion**

\[ T(M, g^M, h^E) := \exp \left( \frac{1}{2} \sum_{p=0}^{m} (-1)^{p+1} \cdot p \cdot \text{Det} \Delta_p \right). \]

- **Ray-Singer torsion**

\[ \rho^{RS}(\nabla) := \rho(\nabla, g^M) \cdot T(M, g^M, h^E) \in \det H^\bullet(M, E). \]

- **Ray-Singer metric** \( \| \cdot \|_{\text{det} H^\bullet(M, E)}^{RS} \) on \( \text{det} H^\bullet(M, E) \)

\[ \| \cdot \|_{\text{det} H^\bullet(M, E)}^{RS} := \| \cdot \|_{\text{det} H^{\bullet}(M, E)}^{L^2} \cdot T(M, g^M, h^E)^{-1}. \]
Ray-Singer analytic torsion

- **Scalar Ray-Singer torsion**

  \[ T(M, g^M, h^E) := \exp \left( \frac{1}{2} \sum_{p=0}^{m} (-1)^{p+1} \cdot p \cdot \text{Det} \Delta_p \right) \].

- **Ray-Singer torsion**

  \[ \rho^{\text{RS}}(\nabla) := \rho(\nabla, g^M) \cdot T(M, g^M, h^E) \in \text{det} H^\bullet(M, E). \]

- **Ray-Singer metric** \( \| \cdot \|^{\text{RS}}_{\text{det} H^\bullet(M, E)} \) on \( \text{det} H^\bullet(M, E) \)

  \[ \| \cdot \|^{\text{RS}}_{\text{det} H^\bullet(M, E)} := \| \cdot \|^{L^2}_{\text{det} H^\bullet(M, E)} \cdot T(M, g^M, h^E)^{-1} \]
Scalar Ray-Singer torsion

\[ T(M, g^M, h^E) := \exp \left( \frac{1}{2} \sum_{p=0}^{m} (-1)^{p+1} \cdot p \cdot \text{Det} \Delta_p \right). \]

Ray-Singer torsion

\[ \rho^{\text{RS}}(\nabla) := \rho(\nabla, g^M) \cdot T(M, g^M, h^E) \in \text{det} H^\bullet(M, E). \]

Ray-Singer metric \( \| \cdot \|^{\text{RS}}_{\text{det} H^\bullet(M,E)} \) on \( \text{det} H^\bullet(M, E) \)

\[ \| \cdot \|^{\text{RS}}_{\text{det} H^\bullet(M,E)} := \| \cdot \|_{\text{det} H^\bullet(M,E)}^{L^2} \cdot T(M, g^M, h^E)^{-1} \]
Cheeger-Müller theorem

- If $\dim M$ odd, $\| \cdot \|_{\det H^\bullet (M,E)}^{\text{RS}}$ does not depend on $g^M, h^E$ a topological invariant.

- If $\dim M$ even, $M$ orientable, $h^E$ flat, then $T(M, g^M, h^E) = 1$.

- If $\dim M$ even, $h^E$ unimodular ($\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$), then $\| \cdot \|_{\det H^\bullet (M,E)}^{\text{RS}}$ does not depend on $g^M$, a topological invariant.

Ray-Singer conjecture: The Ray-Singer torsion coincides with the Reidemeister torsion.

- $h^E$ flat, Cheeger(1978), Müller(1978), RS conj. holds

- $\dim M$ odd, $E$ unimodular, Müller(1991) RS conj. holds

- General case, Bismut-Zhang(1991) RS conj. holds
Cheeger-Müller theorem

- If $\dim M$ odd, $\| \cdot \|_{\det H^\bullet(M,E)}^{\text{RS}}$ does not depend on $g^M, h^E$ a topological invariant.

- If $\dim M$ even, $M$ orientable, $h^E$ flat, then $T(M, g^M, h^E) = 1$.

- If $\dim M$ even, $h^E$ unimodular ($\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$), then $\| \cdot \|_{\det H^\bullet(M,E)}^{\text{RS}}$ does not depend on $g^M$, a topological invariant.

Ray-Singer conjecture: The Ray-Singer torsion coincides with the Reidemeister torsion.

- $h^E$ flat, Cheeger(1978), Müller(1978), RS conj. holds

- $\dim M$ odd, $E$ unimodular, Müller(1991) RS conj. holds

- General case, Bismut-Zhang(1991) RS conj. holds
Cheeger-Müller theorem

- If $\dim M$ odd, $\| \cdot \|_{\det H^\bullet (M,E)}^{\text{RS}}$ does not depend on $g^M$, $h^E$ a topological invariant.

- If $\dim M$ even, $M$ orientable, $h^E$ flat, then $T(M, g^M, h^E) = 1$.

- If $\dim M$ even, $h^E$ unimodular ($\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$), then $\| \cdot \|_{\det H^\bullet (M,E)}^{\text{RS}}$ does not depend on $g^M$, a topological invariant.

Ray-Singer conjecture: The Ray-Singer torsion coincides with the Reidemeister torsion.

- $h^E$ flat, Cheeger(1978), Müller(1978), RS conj. holds

- $\dim M$ odd, $E$ unimodular, Müller(1991) RS conj. holds

- General case, Bismut-Zhang(1991) RS conj. holds
Cheeger-Müller theorem

- If \( \dim M \) odd, \( \| \cdot \|_{\det H^\bullet(M,E)}^{RS} \) does not depend on \( g^M, h^E \) a topological invariant.

- If \( \dim M \) even, \( M \) orientable, \( h^E \) flat, then \( T(M, g^M, h^E) = 1 \).

- If \( \dim M \) even, \( h^E \) unimodular ( \( \det \rho(\gamma) = 1 \) for all \( \gamma \in \pi_1(M) \)) , then \( \| \cdot \|_{\det H^\bullet(M,E)}^{RS} \) does not depend on \( g^M \), a topological invariant.

Ray-Singer conjecture: The Ray-Singer torsion coincides with the Reidemeister torsion.

- \( h^E \) flat, Cheeger(1978), Müller(1978), RS conj. holds

- \( \dim M \) odd, \( E \) unimodular, Müller(1991) RS conj. holds

- General case, Bismut-Zhang(1991) RS conj. holds
Cheeger-Müller theorem

- If $\dim M$ odd, $\| \cdot \|_{\det H^\bullet (M,E)}^{\mathrm{RS}}$ does not depend on $g^M, h^E$ a topological invariant.

- If $\dim M$ even, $M$ orientable, $h^E$ flat, then $T(M, g^M, h^E) = 1$.

- If $\dim M$ even, $h^E$ unimodular ($\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$), then $\| \cdot \|_{\det H^\bullet (M,E)}^{\mathrm{RS}}$ does not depend on $g^M$, a topological invariant.

Ray-Singer conjecture: The Ray-Singer torsion coincides with the Reidemeister torsion.

- $h^E$ flat, Cheeger(1978), Müller(1978), RS conj. holds

- $\dim M$ odd, $E$ unimodular, Müller(1991) RS conj. holds

- General case, Bismut-Zhang(1991) RS conj. holds
Cheeger-Müller theorem

- If $\dim M$ odd, $\| \cdot \|_{\det H^\bullet(M,E)}^{RS}$ does not depend on $g^M, h^E$ a topological invariant.

- If $\dim M$ even, $M$ orientable, $h^E$ flat, then $T(M, g^M, h^E) = 1$.

- If $\dim M$ even, $h^E$ unimodular ($\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$), then $\| \cdot \|_{\det H^\bullet(M,E)}^{RS}$ does not depend on $g^M$, a topological invariant.

Ray-Singer conjecture: The Ray-Singer torsion coincides with the Reidemeister torsion.

- $h^E$ flat, Cheeger(1978), Müller(1978), RS conj. holds

- $\dim M$ odd, $E$ unimodular, Müller(1991) RS conj. holds

- General case, Bismut-Zhang(1991) RS conj. holds
Cheeger-Müller theorem

- If $\text{dim } M$ odd, $\| \cdot \|_{\det H^\bullet(M,E)}^{\text{RS}}$ does not depend on $g^M$, $h^E$ a topological invariant.

- If $\text{dim } M$ even, $M$ orientable, $h^E$ flat, then $T(M, g^M, h^E) = 1$.

- If $\text{dim } M$ even, $h^E$ unimodular ($\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$), then $\| \cdot \|_{\det H^\bullet(M,E)}^{\text{RS}}$ does not depend on $g^M$, a topological invariant.

Ray-Singer conjecture: The Ray-Singer torsion coincides with the Reidemeister torsion.

- $h^E$ flat, Cheeger(1978), Müller(1978), RS conj. holds

- $\text{dim } M$ odd, $E$ unimodular, Müller(1991) RS conj. holds

- General case, Bismut-Zhang(1991) RS conj. holds
Impose relative and absolute boundary conditions for $\triangle$.


- $h^E$ flat, but without assuming product structure near $\partial M$: Dai-Fang(2000)

- Most general case: Brüning-Ma(2006)

- Gluing formula for analytic torsion: Brüning-Ma(2013)
Impose **relative** and **absolute** boundary conditions for $\Delta$.


- $h^E$ flat, but without assuming product structure near $\partial M$: Dai-Fang(2000)

- Most general case: Brüning-Ma(2006)

- Gluing formula for analytic torsion: Brüning-Ma(2013)
Impose **relative** and **absolute** boundary conditions for $\Delta$.


- $h^E$ flat, but without assuming product structure near $\partial M$: Dai-Fang(2000)

- Most general case: Brüning-Ma(2006)

- Gluing formula for analytic torsion: Brüning-Ma(2013)
Impose relative and absolute boundary conditions for $\Delta$.


$h^E$ flat, but without assuming product structure near $\partial M$: Dai-Fang(2000)

Most general case: Brüning-Ma(2006)

Gluing formula for analytic torsion: Brüning-Ma(2013)
Impose relative and absolute boundary conditions for $\Delta$.


$h^E$ flat, but without assuming product structure near $\partial M$: Dai-Fang(2000)

Most general case: Brüning-Ma(2006)

Gluing formula for analytic torsion: Brüning-Ma(2013)
Odd signature operator

- \((M^m, g^M)\) a closed oriented Riemannian manifold, \(m = 2r - 1\).
- \((E, \nabla, h^E)\) a complex flat vector bundle over \(M\).
- Define the Chirality operator by
  \[
  \Gamma := i^r (-1)^{\frac{k(k+1)}{2}} \ast : \Omega^k(M, E) \to \Omega^{m-k}(M, E),
  \]
  where \(\ast\) is the Hodge star operator. Then \(\Gamma^2 = \text{Id}\),
- In general \(\nabla^* = \Gamma \nabla' \Gamma\). If \(\nabla\) Hermitian, then \(\nabla^* = \Gamma \nabla \Gamma\).
- The odd signature operator
  \[
  B := \Gamma \nabla + \nabla \Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)
  \]
  not necessarily self-adjoint.
- Note that, if \(\nabla\) Hermitian, \(B^2 = (\Gamma \nabla + \nabla \Gamma)^2 = \Delta\).
(\(M^m, g^M\)) a closed oriented Riemannian manifold, \(m = 2r - 1\).

\((E, \nabla, h^E)\) a complex flat vector bundle over \(M\).

Define the Chirality operator by

\[ \Gamma := i^r (-1)^{\frac{k(k+1)}{2}} : \Omega^k(M, E) \to \Omega^{m-k}(M, E), \]

where \(\ast\) is the Hodge star operator. Then \(\Gamma^2 = \text{Id}\),

In general \(\nabla^* = \Gamma \nabla' \Gamma\). If \(\nabla\) Hermitian, then \(\nabla^* = \Gamma \nabla \Gamma\).

The odd signature operator

\[ \mathcal{B} := \Gamma \nabla + \nabla \Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E) \]

not necessarily self-adjoint.

Note that, if \(\nabla\) Hermitian, \(\mathcal{B}^2 = (\Gamma \nabla + \nabla \Gamma)^2 = \Delta\).
(\(M^m, g^M\)) a closed oriented Riemannian manifold, \(m = 2r - 1\).

\((E, \nabla, h^E)\) a complex flat vector bundle over \(M\).

Define the Chirality operator by

\[
\Gamma := i^r (-1)^{\frac{k(k+1)}{2}} \ast : \Omega^k(M, E) \to \Omega^{m-k}(M, E),
\]

where \(\ast\) is the Hodge star operator. Then \(\Gamma^2 = \text{Id}\),

In general \(\nabla^\ast = \Gamma \nabla' \Gamma\). If \(\nabla\) Hermitian, then \(\nabla^\ast = \Gamma \nabla \Gamma\).

The odd signature operator

\[
\mathcal{B} := \Gamma \nabla + \nabla \Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)
\]

not necessarily self-adjoint.

Note that, if \(\nabla\) Hermitian, \(\mathcal{B}^2 = (\Gamma \nabla + \nabla \Gamma)^2 = \Delta\).
Odd signature operator

- $(M^m, g^M)$ a closed oriented Riemannian manifold, $m = 2r - 1$.
- $(E, \nabla, h^E)$ a complex flat vector bundle over $M$.
- Define the Chirality operator by

$$
\Gamma := i^r ( -1 ) \frac{k(k+1)}{2} : \Omega^k(M, E) \to \Omega^{m-k}(M, E),
$$

where $\ast$ is the Hodge star operator. Then $\Gamma^2 = \text{Id}$,

- In general $\nabla^\ast = \Gamma \nabla \Gamma$. If $\nabla$ Hermitian, then $\nabla^\ast = \Gamma \nabla \Gamma$.
- The odd signature operator

$$
\mathcal{B} := \Gamma \nabla + \nabla \Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)
$$

not necessarily self-adjoint.
- Note that, if $\nabla$ Hermitian, $\mathcal{B}^2 = (\Gamma \nabla + \nabla \Gamma)^2 = \Delta$. 
Odd signature operator

- \((M^m, g^M)\) a closed oriented Riemannian manifold, \(m = 2r - 1\).
- \((E, \nabla, h^E)\) a complex flat vector bundle over \(M\).
- Define the Chirality operator by
  \[
  \Gamma := i^r (-1)^{\frac{k(k+1)}{2}} \ast : \Omega^k(M, E) \to \Omega^{m-k}(M, E),
  \]
  where \(\ast\) is the Hodge star operator. Then \(\Gamma^2 = \text{Id}\).
- In general \(\nabla^* = \Gamma \nabla' \Gamma\). If \(\nabla\) Hermitian, then \(\nabla^* = \Gamma \nabla \Gamma\).
- The odd signature operator
  \[
  B := \Gamma \nabla + \nabla \Gamma : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)
  \]
  not necessarily self-adjoint.
- Note that, if \(\nabla\) Hermitian, \(B^2 = (\Gamma \nabla + \nabla \Gamma)^2 = \Delta\).
Odd signature operator

- $(M^m, g^M)$ a closed oriented Riemannian manifold, $m = 2r - 1$.
- $(E, \nabla, h^E)$ a complex flat vector bundle over $M$.
- Define the **Chirality operator** by
  
  \[ \Gamma := i^r (-1)^{\frac{k(k+1)}{2}} \star : \Omega^k(M, E) \rightarrow \Omega^{m-k}(M, E), \]

  where $\star$ is the Hodge star operator. Then $\Gamma^2 = \text{Id}$.
- In general $\nabla^* = \Gamma \nabla' \Gamma$. If $\nabla$ Hermitian, then $\nabla^* = \Gamma \nabla \Gamma$.
- The **odd signature operator**

  \[ \mathcal{B} := \Gamma \nabla + \nabla \Gamma : \Omega^\bullet(M, E) \rightarrow \Omega^\bullet(M, E) \]

  is not necessarily self-adjoint.
- Note that, if $\nabla$ Hermitian, $\mathcal{B}^2 = (\Gamma \nabla + \nabla \Gamma)^2 = \Delta$. 

Rung-Tzung Huang (NCU)  
Refined Analytic Torsion  
Workshop on Geom. quant.
Denote by $\Omega^p_+(M, E) = \text{Ker}(\nabla \Gamma) \cap \Omega^p(M, E)$,

$$\Omega^p_-(M, E) = \text{Ker}(\Gamma \nabla) \cap \Omega^p(M, E).$$

**Definition (Braverman-Kapper)**
The graded determinant of $\mathcal{B}_{\text{even}}$ is defined as

$$\text{Det}_{gr}(\mathcal{B}_{\text{even}}) := \frac{\text{Det}(\mathcal{B}|_{\Omega^\text{even}_+(M,E)})}{\text{Det}(-\mathcal{B}|_{\Omega^\text{even}_-(M,E)})}.$$
Graded determinant of $B_{\text{even}}$

- Denote by $\Omega_+^p(M, E) = \text{Ker}(\nabla \Gamma) \cap \Omega^p(M, E)$,

$$\Omega_-^p(M, E) = \text{Ker}(\Gamma \nabla) \cap \Omega^p(M, E).$$

**Definition (Braverman-Kapper)**

The **graded determinant** of $B_{\text{even}}$ is defined as

$$\text{Det}_{gr}(B_{\text{even}}) := \frac{\text{Det}(B|_{\Omega_{\text{even}}^+(M, E)})}{\text{Det}(-B|_{\Omega_{\text{even}}^-(M, E)})}.$$
The $\eta$-function of $B_{\text{even}}$ is defined as

$$\eta(s, B_{\text{even}}) = \sum_{\text{Re } \lambda > 0} \lambda^{-s} - \sum_{\text{Re } \lambda < 0} (-\lambda)^{-s}.$$ 

$\eta(s, B_{\text{even}})$ holomorphic for Re $s$ large and admits a meromorphic extension to $\mathbb{C}$. In particular, $s = 0$ is a regular point.

The $\eta$-invariant of $B_{\text{even}}$ is defined as

$$\eta(B_{\text{even}}) = \frac{\eta(0, B_{\text{even}}) + m_+(B_{\text{even}}) - m_-(B_{\text{even}})}{2},$$

where $m_\pm$ are # of eigenvalues on $\pm$-parts of imaginary axis.

Denote by $\eta_{\text{trivial}}(g^M)$ the $\eta$-invariant for $M \times \mathbb{C}$.
The \( \eta \)-function of \( \mathcal{B}_{\text{even}} \) is defined as

\[
\eta(s, \mathcal{B}_{\text{even}}) = \sum_{\text{Re } \lambda > 0} \lambda^{-s} - \sum_{\text{Re } \lambda < 0} (-\lambda)^{-s}.
\]

\( \eta(s, \mathcal{B}_{\text{even}}) \) holomorphic for \( \text{Re } s \) large and admits a meromorphic extension to \( \mathbb{C} \). In particular, \( s = 0 \) is a regular point.

The \( \eta \)-invariant of \( \mathcal{B}_{\text{even}} \) is defined as

\[
\eta(\mathcal{B}_{\text{even}}) = \frac{\eta(0, \mathcal{B}_{\text{even}}) + m_+(\mathcal{B}_{\text{even}}) - m_-(\mathcal{B}_{\text{even}})}{2},
\]

where \( m_\pm \) are \# of eigenvalues on \( \pm \)-parts of imaginary axis.

Denote by \( \eta_{\text{trivial}}(g^M) \) the \( \eta \)-invariant for \( M \times \mathbb{C} \).
The \( \eta \)-function of \( B_{\text{even}} \) is defined as

\[
\eta(s, B_{\text{even}}) = \sum_{\Re \lambda > 0} \lambda^{-s} - \sum_{\Re \lambda < 0} (-\lambda)^{-s}.
\]

\( \eta(s, B_{\text{even}}) \) holomorphic for \( \Re s \) large and admits a meromorphic extension to \( \mathbb{C} \). In particular, \( s = 0 \) is a regular point.

The \( \eta \)-invariant of \( B_{\text{even}} \) is defined as

\[
\eta(B_{\text{even}}) = \frac{\eta(0, B_{\text{even}}) + m_+(B_{\text{even}}) - m_-(B_{\text{even}})}{2},
\]

where \( m_\pm \) are \# of eigenvalues on \( \pm \)-parts of imaginary axis.

Denote by \( \eta_{\text{trivial}}(g^M) \) the \( \eta \)-invariant for \( M \times \mathbb{C} \).
The $\eta$-function of $\mathcal{B}_{\text{even}}$ is defined as

$$
\eta(s, \mathcal{B}_{\text{even}}) = \sum_{\text{Re } \lambda > 0} \lambda^{-s} - \sum_{\text{Re } \lambda < 0} (-\lambda)^{-s}.
$$

$\eta(s, \mathcal{B}_{\text{even}})$ holomorphic for $\text{Re } s$ large and admits a meromorphic extension to $\mathbb{C}$. In particular, $s = 0$ is a regular point.

The $\eta$-invariant of $\mathcal{B}_{\text{even}}$ is defined as

$$
\eta(\mathcal{B}_{\text{even}}) = \frac{\eta(0, \mathcal{B}_{\text{even}}) + m_{+}(\mathcal{B}_{\text{even}}) - m_{-}(\mathcal{B}_{\text{even}})}{2},
$$

where $m_{\pm}$ are $\#$ of eigenvalues on $\pm$-parts of imaginary axis.

Denote by $\eta_{\text{trivial}}(g^{M})$ the $\eta$-invariant for $M \times \mathbb{C}$.
Relation with $\eta$-invariant

**Proposition**

If

$$\xi = \frac{1}{2} \sum_{p=0}^{m} (-1)^{p+1} \cdot p \cdot \log \Det B^2 |_{\Omega^p(M,E)}$$

then

$$\Det_{gr}(B_{\text{even}}) = e^{\xi - i\pi(\eta(B_{\text{even}}) + \cdots)}.$$ 

In particular, if $\nabla$ is acyclic (i.e. $H^\bullet(M,E) = 0$) and Hermitian, then

$$\log \Det_{gr}(B_{\text{even}}) = \log \rho^{RS}(\nabla) - i\pi \eta(B_{\text{even}}).$$
Relation with $\eta$-invariant

Proposition

If

$$\xi = \frac{1}{2} \sum_{p=0}^{m} (-1)^{p+1} \cdot p \cdot \log \operatorname{Det} B^2|_{\Omega^p(M,E)}$$

then

$$\operatorname{Det}_{\operatorname{gr}}(B_{\text{even}}) = e^{\xi - i\pi \eta(B_{\text{even}}) + \cdots}.$$  

In particular, if $\nabla$ is acyclic (i.e. $H^\bullet(M, E) = 0$) and Hermitian, then

$$\log \operatorname{Det}_{\operatorname{gr}}(B_{\text{even}}) = \log \rho_{\text{RS}}(\nabla) - i\pi \eta(B_{\text{even}}).$$
Refined analytic torsion

Volume element:

\[ \rho_\Gamma(\nabla, g^M) = (-1)^R \cdot [h_0] \otimes [h_1]^{-1} \otimes \cdots \otimes [h_{r-1}]^{(-1)^{r-1}} \]

\[ \otimes [\Gamma h_{r-1}]^{(-1)^r} \otimes [\Gamma h_{r-2}]^{(-1)^{r-1}} \otimes \cdots \otimes [\Gamma h_0]^{-1} \]

and \( R \) an algebraic formula on Betti numbers \( \beta_p(M, E) \)

**Definition & Theorem (Braverman-Kappeler 2007)**

The refined analytic torsion defined by

\[ \rho_{an}(\nabla) := \rho_\Gamma(\nabla, g^M) \cdot \text{Det}_{gr}(\mathcal{B}_{\text{even}}) \cdot e^{i\pi \cdot \text{rk} E \cdot \eta_{\text{trivial}}(g^M)}, \]

independent of the choice of \( g^M \) and a topological invariant.
Refined analytic torsion

Volume element:

\[
\rho_{\Gamma}(\nabla, g^M) = (-1)^R \cdot [h_0] \otimes [h_1]^{-1} \otimes \cdots \otimes [h_{r-1}]^{(-1)^{r-1}} \otimes [\Gamma h_{r-1}]^{(-1)^{r}} \otimes [\Gamma h_{r-2}]^{(-1)^{r-1}} \otimes \cdots \otimes [\Gamma h_0]^{(-1)}
\]

and \( R \) an algebraic formula on Betti numbers \( \beta_p(M, E) \)

Definition & Theorem (Braverman-Kappeler 2007)

The refined analytic torsion defined by

\[
\rho_{an}(\nabla) := \rho_{\Gamma}(\nabla, g^M) \cdot \text{Det}_{\text{gr}}(\mathcal{B}_{\text{even}}) \cdot e^{i\pi \cdot \text{rk} E \cdot \eta_{\text{trivial}}(g^M)},
\]

independent of the choice of \( g^M \) and a topological invariant.
Some properties of refined analytic torsion

- **(Braverman-Kappeler 2007)**: The refined analytic torsion is closely related to the Farber-Turaev torsion, a refinement of the Reidemeister torsion.

- If $E$ acyclic and $\nabla$ Hermitian, then $|\rho_{\text{an}}(\nabla)| = \rho^{\text{RS}}(\nabla)$ and $\text{Ph}(\rho_{\text{an}}(\nabla)) = -\pi \rho(\nabla)$, where $\rho(\nabla) = \eta(B_{\text{even}}) - \text{rank} E \cdot \eta_{\text{trivial}}(g^M)$.

- Refined analytic torsion is an analytic function on the space of representation variety.

- **Braverman-Vertman (2013)**: An alternative derivation of the Bismut-Zhang’s formula on the connected components of the complex representation space which contain a unitary point.
Some properties of refined analytic torsion

(Braverman-Kappeler 2007): The refined analytic torsion is closely related to the Farber-Turaev torsion, a refinement of the Reidemeister torsion.

If $E$ acyclic and $\nabla$ Hermitian, then $|\rho_{an}(\nabla)| = \rho^{RS}(\nabla)$ and $\text{Ph}(\rho_{an}(\nabla)) = -\pi \rho(\nabla)$, where $\rho(\nabla) = \eta(B_{\text{even}}) - \text{rank} E \cdot \eta_{\text{trivial}}(g^M)$.

Refined analytic torsion is an analytic function on the space of representation variety.

Braverman-Vertman (2013): An alternative derivation of the Bismut-Zhang’s formula on the connected components of the complex representation space which contain a unitary point.
Some properties of refined analytic torsion

- **(Braverman-Kappeler 2007):** The refined analytic torsion is closely related to the Farber-Turaev torsion, a refinement of the Reidemeister torsion.

- If $E$ acyclic and $\nabla$ Hermitian, then $|\rho_{an}(\nabla)| = \rho_{RS}(\nabla)$ and $\text{Ph}(\rho_{an}(\nabla)) = -\pi \rho(\nabla)$, where $\rho(\nabla) = \eta(B_{even}) - \text{rank } E \cdot \eta_{\text{trivial}}(g^{M})$.

- Refined analytic torsion is an analytic function on the space of representation variety.

- **Braverman-Vertman (2013):** An alternative derivation of the Bismut-Zhang’s formula on the connected components of the complex representation space which contain a unitary point.
Some properties of refined analytic torsion

- **(Braverman-Kappeler 2007)**: The refined analytic torsion is closely related to the Farber-Turaev torsion, a refinement of the Reidemeister torsion.

- If $E$ acyclic and $\nabla$ Hermitian, then $|\rho_{an}(\nabla)| = \rho^{RS}(\nabla)$ and $\text{Ph}(\rho_{an}(\nabla)) = -\pi \rho(\nabla)$, where $\rho(\nabla) = \eta(B_{\text{even}}) - \text{rank} E \cdot \eta_{\text{trivial}}(g^M)$.

- Refined analytic torsion is an analytic function on the space of representation variety.

- **Braverman-Vertman (2013)**: An alternative derivation of the Bismut-Zhang’s formula on the connected components of the complex representation space which contain a unitary point.
When $\partial M \neq \emptyset$, Vertman(2009) and Huang-Lee(2010) in two different independent constructions.

- Braverman-Vertman (2015): A gluing formula for refined analytic torsion on connected components of the complex representation space which contain a unitary point.

- The essential ingredient in the definition of the refined analytic torsion is the twisted de Rham complex with a chirality operator and the odd signature operator associated to the complex.

- Roughly speaking, Vertman considers

$$
\Omega^\bullet_{\text{rel}}(M, E) \oplus \Omega^\bullet_{\text{abs}}(M, E), \quad \tilde{\Gamma} = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}, \quad \tilde{B}_{\text{even}} := \begin{pmatrix} 0 & B_{\text{even}} \\ B_{\text{even}} & 0 \end{pmatrix}
$$
When $\partial M \neq \emptyset$, Vertman (2009) and Huang-Lee (2010) in two different independent constructions.

Braverman-Vertman (2015): A gluing formula for refined analytic torsion on connected components of the complex representation space which contain a unitary point.

The essential ingredient in the definition of the refined analytic torsion is the twisted de Rham complex with a chirality operator and the odd signature operator associated to the complex.

Roughly speaking, Vertman considers

$$\Omega_{\text{rel}}(M, E) \oplus \Omega_{\text{abs}}(M, E), \quad \tilde{\Gamma} = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}, \quad \tilde{B}_{\text{even}} := \begin{pmatrix} 0 & B_{\text{even}} \\ B_{\text{even}} & 0 \end{pmatrix}$$
When $\partial M \neq \emptyset$, Vertman (2009) and Huang-Lee (2010) in two different independent constructions.

Braverman-Vertman (2015): A gluing formula for refined analytic torsion on connected components of the complex representation space which contain a unitary point.

The essential ingredient in the definition of the refined analytic torsion is the twisted de Rham complex with a chirality operator and the odd signature operator associated to the complex.

Roughly speaking, Vertman considers

$$
\Omega^\bullet_{\text{rel}}(M, E) \oplus \Omega^\bullet_{\text{abs}}(M, E), \quad \tilde{\Gamma} = \begin{pmatrix}
0 & \Gamma \\
\Gamma & 0
\end{pmatrix}, \quad \tilde{B}_{\text{even}} := \begin{pmatrix}
0 & B_{\text{even}} \\
B_{\text{even}} & 0
\end{pmatrix}
$$
When $\partial M \neq \emptyset$, Vertman(2009) and Huang-Lee(2010) in two different independent constructions.

Braverman-Vertman (2015): A gluing formula for refined analytic torsion on connected components of the complex representation space which contain a unitary point.

The essential ingredient in the definition of the refined analytic torsion is the twisted de Rham complex with a chirality operator and the odd signature operator associated to the complex.

Roughly speaking, Vertman considers

$$\Omega^\bullet_{\text{rel}}(M, E) \oplus \Omega^\bullet_{\text{abs}}(M, E), \quad \widetilde{\Gamma} = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}, \quad \widetilde{B}_{\text{even}} := \begin{pmatrix} 0 & B_{\text{even}} \\ B_{\text{even}} & 0 \end{pmatrix}$$
Vertman’s approach

**Lemma**

$\text{Spec}(\tilde{B}_{\text{even},\text{rel/abs}})$ is symmetric w.r.t. 0. Hence $\eta\left(\tilde{B}_{\text{even},\text{rel/abs}}\right) = 0$.

**Proof.**

\[
\begin{pmatrix}
0 & \mathcal{B}_{\text{even}} \\
\mathcal{B}_{\text{even}} & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{\text{rel}} \\
\psi_{\text{abs}}
\end{pmatrix}
= \begin{pmatrix}
\mathcal{B}_{\text{even}} \psi_{\text{abs}} \\
\mathcal{B}_{\text{even}} \phi_{\text{rel}}
\end{pmatrix}
= \lambda \begin{pmatrix}
\phi_{\text{rel}} \\
\psi_{\text{abs}}
\end{pmatrix}.
\]

\[
\begin{pmatrix}
0 & \mathcal{B}_{\text{even}} \\
\mathcal{B}_{\text{even}} & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{\text{rel}} \\
-\psi_{\text{abs}}
\end{pmatrix}
= \begin{pmatrix}
-\mathcal{B}_{\text{even}} \psi_{\text{abs}} \\
\mathcal{B}_{\text{even}} \phi_{\text{rel}}
\end{pmatrix}
= -\lambda \begin{pmatrix}
\phi_{\text{rel}} \\
-\psi_{\text{abs}}
\end{pmatrix}.
\]

**Proposition**

If $\nabla$ is acyclic and Hermitian, then

$$\log \text{Det}_{\text{gr}} \tilde{B}_{\text{even,rel/abs}} = (\log \rho_{\text{rel}}^{\text{RS}}(\nabla) + \log \rho_{\text{abs}}^{\text{RS}}(\nabla)).$$
Vertman’s approach

**Lemma**

$\text{Spec}(\tilde{B}_{\text{even,rel/abs}})$ is symmetric w.r.t. 0. Hence $\eta(\tilde{B}_{\text{even,rel/abs}}) = 0$.

**Proof.**

\[
\begin{pmatrix}
0 & B_{\text{even}} \\
B_{\text{even}} & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{\text{rel}} \\
\psi_{\text{abs}}
\end{pmatrix}
= 
\begin{pmatrix}
B_{\text{even}} \psi_{\text{abs}} \\
B_{\text{even}} \phi_{\text{rel}}
\end{pmatrix}
= \lambda
\begin{pmatrix}
\phi_{\text{rel}} \\
\psi_{\text{abs}}
\end{pmatrix}.
\]

\[
\begin{pmatrix}
0 & B_{\text{even}} \\
B_{\text{even}} & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{\text{rel}} \\
-\psi_{\text{abs}}
\end{pmatrix}
= 
\begin{pmatrix}
-B_{\text{even}} \psi_{\text{abs}} \\
B_{\text{even}} \phi_{\text{rel}}
\end{pmatrix}
= -\lambda
\begin{pmatrix}
\phi_{\text{rel}} \\
-\psi_{\text{abs}}
\end{pmatrix}.
\]

**Proposition**

If $\nabla$ is acyclic and Hermitian, then

\[
\log \text{Det}_{\text{gr}} \tilde{B}_{\text{even,rel/abs}} = (\log \rho_{\text{rel}}^{\text{RS}}(\nabla) + \log \rho_{\text{abs}}^{\text{RS}}(\nabla)).
\]
Vertman’s approach

Lemma

$\text{Spec}(\tilde{\mathcal{B}}_{\text{even},\text{rel}/\text{abs}})$ is symmetric w.r.t. 0. Hence $\eta(\tilde{\mathcal{B}}_{\text{even},\text{rel}/\text{abs}}) = 0$.

Proof.

$$
\begin{pmatrix}
0 & \mathcal{B}_{\text{even}} \\
\mathcal{B}_{\text{even}} & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{\text{rel}} \\
\psi_{\text{abs}}
\end{pmatrix}
= 
\begin{pmatrix}
\mathcal{B}_{\text{even}} \psi_{\text{abs}} \\
\mathcal{B}_{\text{even}} \phi_{\text{rel}}
\end{pmatrix}
= \lambda
\begin{pmatrix}
\phi_{\text{rel}} \\
\psi_{\text{abs}}
\end{pmatrix}.
$$

$$
\begin{pmatrix}
0 & \mathcal{B}_{\text{even}} \\
\mathcal{B}_{\text{even}} & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{\text{rel}} \\
-\psi_{\text{abs}}
\end{pmatrix}
= 
\begin{pmatrix}
-\mathcal{B}_{\text{even}} \psi_{\text{abs}} \\
\mathcal{B}_{\text{even}} \phi_{\text{rel}}
\end{pmatrix}
= -\lambda
\begin{pmatrix}
\phi_{\text{rel}} \\
-\psi_{\text{abs}}
\end{pmatrix}.
$$

Proposition

If $\nabla$ is acyclic and Hermitian, then

$$
\log \text{Det}_{\text{gr}} \tilde{\mathcal{B}}_{\text{even},\text{rel}/\text{abs}} = (\log \rho^{\text{RS}}_{\text{rel}}(\nabla) + \log \rho^{\text{RS}}_{\text{abs}}(\nabla))
$$
Boundary conditions for $\mathcal{B}$

- Now assume $\partial M = Y \neq \phi$ and $g^M$ is a product metric near $Y$.
- Trivialize $E$ along the normal direction near $Y$ by using $\nabla$.
- Assume $\nabla$ is Hermitian.
- For $\phi \in \Omega^\bullet(M, E)$ and $\mathcal{B}\phi = 0$, near $Y$,
  $$\phi = \nabla^Y \phi_1 + \phi_2 + du \wedge ((\nabla^Y)^* \psi_1 + \psi_2),$$
  where $\phi_2, \psi_2 \in \text{Ker} \, \Delta_Y$.
- We define
  $$\mathcal{K} = \{ \phi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}, \quad \Gamma^Y \mathcal{K} = \{ \psi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}.$$  
- Then
  $$\mathcal{K} \cong \text{Im} (\iota^* : H^\bullet (M, E) \to H^\bullet (Y, E|_Y)), \quad \text{where } \iota : Y \hookrightarrow M$$
  and
  $$H^\bullet (Y, E|_Y) \cong \text{Ker} \, \Delta_Y = \mathcal{K} \oplus \Gamma^Y \mathcal{K}.$$
Boundary conditions for $B$

- Now assume $\partial M = Y \neq \phi$ and $g^M$ is a product metric near $Y$.
- Trivialize $E$ along the normal direction near $Y$ by using $\nabla$.
- Assume $\nabla$ is Hermitian.
- For $\phi \in \Omega^\bullet(M, E)$ and $B\phi = 0$, near $Y$,
  \[ \phi = \nabla^Y \phi_1 + \phi_2 + du \wedge ((\nabla^Y)^* \psi_1 + \psi_2), \]
  where $\phi_2, \psi_2 \in \text{Ker } \Delta_Y$.
- We define
  \[ K = \{ \phi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}, \quad \Gamma^Y K = \{ \psi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}. \]
- Then
  \[ K \cong \text{Im}(\iota^* : H^\bullet(M, E) \rightarrow H^\bullet(Y, E|_Y)), \]
  where $\iota : Y \hookrightarrow M$
  and
  \[ H^\bullet(Y, E|_Y) \cong \text{Ker } \Delta_Y = K \oplus \Gamma^Y K. \]
Boundary conditions for $\mathcal{B}$

- Now assume $\partial M = Y \neq \phi$ and $g^M$ is a product metric near $Y$.
- Trivialize $E$ along the normal direction near $Y$ by using $\nabla$.
- Assume $\nabla$ is Hermitian.

- For $\phi \in \Omega^\bullet(M, E)$ and $\mathcal{B}\phi = 0$, near $Y$,
  $$\phi = \nabla^Y \phi_1 + \phi_2 + du \wedge ((\nabla^Y)^* \psi_1 + \psi_2),$$
  where $\phi_2, \psi_2 \in \text{Ker } \Delta_Y$.

- We define
  $$\mathcal{K} = \{ \phi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}, \quad \Gamma^Y \mathcal{K} = \{ \psi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}.$$

- Then
  $$\mathcal{K} \cong \text{Im}(\iota^* : H^\bullet(M, E) \to H^\bullet(Y, E|_Y)), \quad \text{where } \iota : Y \hookrightarrow M$$
  and
  $$H^\bullet(Y, E|_Y) \cong \text{Ker } \Delta_Y = \mathcal{K} \oplus \Gamma^Y \mathcal{K}.$$
Boundary conditions for $\mathcal{B}$

- Now assume $\partial M = Y \neq \phi$ and $g^M$ is a product metric near $Y$.
- Trivialize $E$ along the normal direction near $Y$ by using $\nabla$.
- Assume $\nabla$ is Hermitian.
- For $\phi \in \Omega^\bullet(M, E)$ and $\mathcal{B}\phi = 0$, near $Y$,
  \[
  \phi = \nabla^Y \phi_1 + \phi_2 + du \wedge ((\nabla^Y)^* \psi_1 + \psi_2),
  \]
  where $\phi_2, \psi_2 \in \text{Ker} \Delta_Y$.

- We define
  \[
  \mathcal{K} = \{\phi_2 | \nabla \phi = \Gamma \nabla \Gamma \phi = 0\}, \quad \Gamma^Y \mathcal{K} = \{\psi_2 | \nabla \phi = \Gamma \nabla \Gamma \phi = 0\}.
  \]
- Then
  \[
  \mathcal{K} \cong \text{Im}(\iota^* : H^\bullet(M, E) \to H^\bullet(Y, E|_Y)), \quad \text{where } \iota : Y \hookrightarrow M
  \]
  and
  \[
  H^\bullet(Y, E|_Y) \cong \text{Ker} \Delta_Y = \mathcal{K} \oplus \Gamma^Y \mathcal{K}.
  \]
Boundary conditions for $\mathcal{B}$

- Now assume $\partial M = Y \neq \phi$ and $g^M$ is a product metric near $Y$.
- Trivialize $E$ along the normal direction near $Y$ by using $\nabla$.
- Assume $\nabla$ is Hermitian.
- For $\phi \in \Omega^\bullet(M, E)$ and $\mathcal{B}\phi = 0$, near $Y$,
  \[ \phi = \nabla^Y \phi_1 + \phi_2 + du \wedge ((\nabla^Y)^* \psi_1 + \psi_2), \]
  where $\phi_2, \psi_2 \in \text{Ker} \Delta_Y$.
- We define
  \[ \mathcal{K} = \{ \phi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}, \quad \Gamma^Y \mathcal{K} = \{ \psi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}. \]
- Then
  \[ \mathcal{K} \cong \text{Im}(\iota^* : H^\bullet(M, E) \to H^\bullet(Y, E|_Y)), \quad \text{where } \iota : Y \hookrightarrow M \]
  and
  \[ H^\bullet(Y, E|_Y) \cong \text{Ker} \Delta_Y = \mathcal{K} \oplus \Gamma^Y \mathcal{K}. \]
Boundary conditions for $\mathcal{B}$

Now assume $\partial M = Y \neq \phi$ and $g^M$ is a product metric near $Y$.

Trivialize $E$ along the normal direction near $Y$ by using $\nabla$.

Assume $\nabla$ is Hermitian.

For $\phi \in \Omega^\bullet(M, E)$ and $\mathcal{B}\phi = 0$, near $Y$,

$$\phi = \nabla^Y \phi_1 + \phi_2 + du \wedge (\nabla^Y)^* \psi_1 + \psi_2,$$

where $\phi_2, \psi_2 \in \text{Ker} \Delta_Y$.

We define

$$\mathcal{K} = \{ \phi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}, \quad \Gamma^Y \mathcal{K} = \{ \psi_2 \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \}.$$

Then

$$\mathcal{K} \cong \text{Im}(\iota^* : H^\bullet(M, E) \to H^\bullet(Y, E|_Y)), \quad \text{where } \iota : Y \hookrightarrow M$$

and

$$H^\bullet(Y, E|_Y) \cong \text{Ker} \Delta_Y = \mathcal{K} \oplus \Gamma^Y \mathcal{K}.$$
Projections $\mathcal{P}_-$ and $\mathcal{P}_+$

- **Hodge decomposition:**

$$\Omega^\bullet(Y, E|_Y) = \text{Im} \nabla^Y \oplus \text{Im}(\nabla^Y)^* \oplus \mathcal{K} \oplus \Gamma^Y \mathcal{K}.$$  

- Define the orthogonal projections $\mathcal{P}_-$, $\mathcal{P}_+$ by

$$\text{Im} \mathcal{P}_- = \begin{pmatrix} \text{Im} \nabla^Y \oplus \mathcal{K} \\ \text{Im} \nabla^Y \oplus \mathcal{K} \end{pmatrix}, \quad \text{Im} \mathcal{P}_+ = \begin{pmatrix} \text{Im}(\nabla^Y)^* \oplus \Gamma^Y \mathcal{K} \\ \text{Im}(\nabla^Y)^* \oplus \Gamma^Y \mathcal{K} \end{pmatrix}.$$  

- Define the realization $\mathcal{B}_\mathcal{P}_-$ by $\mathcal{B}$ with domain

$$\text{Dom}(\mathcal{B}_\mathcal{P}_-) = \{ \psi \in \Omega^\bullet(M, E) | \mathcal{P}_-(\psi|_Y) = 0 \},$$

and similarly, for $\mathcal{B}_\mathcal{P}_+$.

- Note that

$$\Gamma^Y \mathcal{P}_- \Gamma^Y = \mathcal{P}_+, \quad \Gamma \mathcal{B}_\mathcal{P}_- \Gamma = \mathcal{B}_\mathcal{P}_+.$$
Projections $\mathcal{P}_-$ and $\mathcal{P}_+$

- **Hodge decomposition:**

$$\Omega^\bullet(Y, E|_Y) = \text{Im } \nabla^Y \oplus \text{Im } (\nabla^Y)^* \oplus K \oplus \Gamma^Y K.$$

- **Define the orthogonal projections $\mathcal{P}_-$, $\mathcal{P}_+$ by**

$$\text{Im } \mathcal{P}_- = \begin{pmatrix} \text{Im } \nabla^Y \oplus K \\ \text{Im } \nabla^Y \oplus K \end{pmatrix}, \quad \text{Im } \mathcal{P}_+ = \begin{pmatrix} \text{Im } (\nabla^Y)^* \oplus \Gamma^Y K \\ \text{Im } (\nabla^Y)^* \oplus \Gamma^Y K \end{pmatrix}.$$

- **Define the realization $\mathcal{B}_{\mathcal{P}_-}$ by $\mathcal{B}$ with domain**

$$\text{Dom}(\mathcal{B}_{\mathcal{P}_-}) = \{ \psi \in \Omega^\bullet(M, E) | \mathcal{P}_-(\psi|_Y) = 0 \},$$

and similarly, for $\mathcal{B}_{\mathcal{P}_+}$.

- **Note that**

$$\Gamma^Y \mathcal{P}_- \Gamma^Y = \mathcal{P}_+, \quad \Gamma \mathcal{B}_{\mathcal{P}_-} \Gamma = \mathcal{B}_{\mathcal{P}_+}.$$
Projections $\mathcal{P}_-$ and $\mathcal{P}_+$

- **Hodge decomposition:**

$$\Omega^\bullet(Y, E|_Y) = \text{Im} \, \nabla^Y \oplus \text{Im} (\nabla^Y)^* \oplus \mathcal{K} \oplus \Gamma^Y \mathcal{K}.$$ 

- Define the orthogonal projections $\mathcal{P}_-$, $\mathcal{P}_+$ by

$$\text{Im} \mathcal{P}_- = \begin{pmatrix} \text{Im} \nabla^Y \oplus \mathcal{K} \\ \text{Im} \nabla^Y \oplus \mathcal{K} \end{pmatrix}, \quad \text{Im} \mathcal{P}_+ = \begin{pmatrix} \text{Im}(\nabla^Y)^* \oplus \Gamma^Y \mathcal{K} \\ \text{Im}(\nabla^Y)^* \oplus \Gamma^Y \mathcal{K} \end{pmatrix}$$

- Define the realization $\mathcal{B}_{\mathcal{P}_-}$ by $\mathcal{B}$ with domain

$$\text{Dom}(\mathcal{B}_{\mathcal{P}_-}) = \{ \psi \in \Omega^\bullet(M, E) | \mathcal{P}_-(\psi|_Y) = 0 \},$$

and similarly, for $\mathcal{B}_{\mathcal{P}_+}$.

- Note that

$$\Gamma^Y \mathcal{P}_- \Gamma^Y = \mathcal{P}_+, \quad \Gamma \mathcal{B}_{\mathcal{P}_-} \Gamma = \mathcal{B}_{\mathcal{P}_+}.$$
Projections $\mathcal{P}_-$ and $\mathcal{P}_+$

- **Hodge decomposition:**
  \[ \Omega^\bullet(Y, E|_Y) = \operatorname{Im} \nabla^Y \oplus \operatorname{Im}(\nabla^Y)^* \oplus \mathcal{K} \oplus \Gamma^Y \mathcal{K}. \]

- Define the orthogonal projections $\mathcal{P}_-$, $\mathcal{P}_+$ by
  \[ \operatorname{Im} \mathcal{P}_- = \begin{pmatrix} \operatorname{Im} \nabla^Y \oplus \mathcal{K} \\ \operatorname{Im} \nabla^Y \oplus \mathcal{K} \end{pmatrix}, \quad \operatorname{Im} \mathcal{P}_+ = \begin{pmatrix} \operatorname{Im}(\nabla^Y)^* \oplus \Gamma^Y \mathcal{K} \\ \operatorname{Im}(\nabla^Y)^* \oplus \Gamma^Y \mathcal{K} \end{pmatrix}. \]

- Define the realization $\mathcal{B}_{\mathcal{P}_-}$ by $\mathcal{B}$ with domain
  \[ \operatorname{Dom}(\mathcal{B}_{\mathcal{P}_-}) = \{ \psi \in \Omega^\bullet(M, E) | \mathcal{P}_-(\psi|_Y) = 0 \}, \]
  and similarly, for $\mathcal{B}_{\mathcal{P}_+}$.

- Note that
  \[ \Gamma^Y \mathcal{P}_- \Gamma^Y = \mathcal{P}_+, \quad \Gamma \mathcal{B}_{\mathcal{P}_-} \Gamma = \mathcal{B}_{\mathcal{P}_+}. \]
A cochain complex with a chirality operator $\Gamma$

- **Cochain complexes** $\left( \Omega_{P_0/1}^\bullet (M, E), \nabla, \Gamma \right)$:

$$0 \to \Omega^0_{P \mp} (M, E) \xrightarrow{\nabla} \Omega^1_{P \pm} (M, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^m_{P \pm} (M, E) \to 0,$$

where

$$\Omega^q_{P \pm} (M, E) := \{ \psi \in \Omega^q (M, E) | \mathcal{P} \pm ( (B^l \psi) |_Y ) = 0, \quad l = 0, 1, 2, \cdots \}$$

**Proposition**

- $H^q_{P_-} (M, E) := H^q (\Omega_{P_-}^\bullet (M, E), \nabla) \cong \text{Ker } B^2_{q, P_-} = \text{Ker } B^2_{q, \text{rel}} \cong H^q (M, Y, E)$,

- $H^q_{P_+} (M, E) := H^q (\Omega_{P_+}^\bullet (M, E), \nabla) \cong \text{Ker } B^2_{q, P_+} = \text{Ker } B^2_{q, \text{abs}} \cong H^q (M, E)$. 

Rung-Tzung Huang (NCU)
A cochain complex with a chirality operator $\Gamma$

- Cochain complexes $\left( \Omega^\bullet_{\mathcal{P}^{0/1}}(M,E), \nabla, \Gamma \right)$:

  
  
  $$0 \to \Omega^0_{\mathcal{P}^\mp}(M,E) \overset{\nabla}{\to} \Omega^1_{\mathcal{P}^\pm}(M,E) \overset{\nabla}{\to} \cdots \overset{\nabla}{\to} \Omega^m_{\mathcal{P}^\pm}(M,E) \to 0,$$

  
  
  where

  
  $$\Omega^q_{\mathcal{P}^\pm}(M,E) := \{ \psi \in \Omega^q(M,E) | \mathcal{P}^\pm ((\mathcal{B}^l \psi)|_Y) = 0, \quad l = 0, 1, 2, \cdots \}$$

Proposition

- $H^q_{\mathcal{P}^-}(M,E) := H^q(\Omega^\bullet_{\mathcal{P}^-}(M,E), \nabla) \cong \text{Ker } \mathcal{B}^2_{q,\mathcal{P}^-} = \text{Ker } \mathcal{B}^2_{q,\text{rel}} \cong H^q(M,Y,E)$,

- $H^q_{\mathcal{P}^+}(M,E) := H^q(\Omega^\bullet_{\mathcal{P}^+}(M,E), \nabla) \cong \text{Ker } \mathcal{B}^2_{q,\mathcal{P}^+} = \text{Ker } \mathcal{B}^2_{q,\text{abs}} \cong H^q(M,E)$.
A cochain complex with a chirality operator $\Gamma$

- Cochain complexes $\left( \Omega^\bullet_{\tilde{P}_{0/1}} (M, E), \nabla, \Gamma \right)$:

$$0 \to \Omega^0_{\tilde{P}^+} (M, E) \xrightarrow{\nabla} \Omega^1_{\tilde{P}^+} (M, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^m_{\tilde{P}^+} (M, E) \to 0,$$

where

$$\Omega^q_{\tilde{P}^\pm} (M, E) := \{ \psi \in \Omega^q (M, E) | \mathcal{P}^\pm ( (\mathcal{B}^l \psi) |_Y) = 0, \ l = 0, 1, 2, \cdots \}$$

Proposition

- $H^q_{\mathcal{P}^-} (M, E) := H^q (\Omega^\bullet_{\mathcal{P}^-} (M, E), \nabla) \cong \text{Ker} \mathcal{B}^2_{\mathcal{P}^-} = \text{Ker} \mathcal{B}^2_{q, \text{rel}} \cong H^q (M, Y, E)$,

- $H^q_{\mathcal{P}^+} (M, E) := H^q (\Omega^\bullet_{\mathcal{P}^+} (M, E), \nabla) \cong \text{Ker} \mathcal{B}^2_{\mathcal{P}^+} = \text{Ker} \mathcal{B}^2_{q, \text{abs}} \cong H^q (M, E)$. 
Refined analytic torsion on manifolds with boundary

**Definition & Theorem (— Y. Lee)**

Under above assumptions. The **refined analytic torsion** defined by

\[
\rho_{an, P^-}(\nabla) := \rho_{\Gamma, P_0}(\nabla, g^M) \cdot \det_{gr}(B_{even, P_-}) \cdot e^{i\pi \cdot \text{rk} E \cdot \eta_{\text{trivial}, P_-}(g^M)},
\]

is independent of the choice of \( g^M \) in the interior of \( M \).

- If \( \nabla \) is acyclic and Hermitian, then

  \[
  \log \det_{gr}(B_{even, P_-}) + \log \det_{gr}(-B_{even, P_+}) = \left( \log \rho_{rel}^{RS}(\nabla) + \log \rho_{abs}^{RS}(\nabla) \right) - i\pi \left( \eta(B_{even, P_-}) - \eta(B_{even, P_+}) \right)
  \]
Under above assumptions. The **refined analytic torsion** defined by

\[
\rho_{an,\mathcal{P}_-}(\nabla) := \rho_{\Gamma,\tilde{\mathcal{P}}_0}(\nabla, g^M) \cdot \text{Det}_{gr}(\mathcal{B}_{\text{even},\mathcal{P}_-}) \cdot e^{i\pi \cdot \text{rk} E \cdot \eta_{\text{trivial},\mathcal{P}_-}(g^M)},
\]

is independent of the choice of \( g^M \) in the interior of \( M \).

- If \( \nabla \) is **acyclic** and **Hermitian**, then

\[
\log \text{Det}_{gr}(\mathcal{B}_{\text{even},\mathcal{P}_-}) + \log \text{Det}_{gr}(-\mathcal{B}_{\text{even},\mathcal{P}_+})
\]

\[
= \left( \log \rho_{\text{rel}}^{\text{RS}}(\nabla) + \log \rho_{\text{abs}}^{\text{RS}}(\nabla) \right) - i\pi \left( \eta(\mathcal{B}_{\text{even},\mathcal{P}_-}) - \eta(\mathcal{B}_{\text{even},\mathcal{P}_+}) \right)
\]
Comparison theorem for refined analytic torsions

- $\hat{\rho}_{\text{an}, P_+}(\nabla)$ refined analytic torsion defined by $-\Gamma$ instead of $\Gamma$.
- The fusion isomorphism

$$\mu : \text{det} H_{\hat{\rho}_0}^\bullet (M, E) \otimes \text{det} H_{\hat{\rho}_1}^\bullet (M, E) \rightarrow \text{det}(H_{\text{rel}}^\bullet (M, E) \oplus H_{\text{abs}}^\bullet (M, E))$$

Theorem (— Y. Lee)

Under above assumptions. Then:

$$\mu \left( \rho_{\text{an}, P_-}(\nabla) \otimes \hat{\rho}_{\text{an}, P_+}(\nabla) \right) = \pm \rho_{\text{an, rel/abs}}(\nabla) \cdot e^{i\pi \frac{\text{rk} E \cdot \chi(M,C)}}.$$
Comparison theorem for refined analytic torsions

- \( \hat{\rho}_{an,P_+}(\nabla) \) refined analytic torsion defined by \(-\Gamma\) instead of \(\Gamma\).
- The fusion isomorphism

\[
\mu : \det H^\bullet_{\mathcal{P}_0}(M, E) \otimes \det H^\bullet_{\mathcal{P}_1}(M, E) \to \det(H^\bullet_{\text{rel}}(M, E) \oplus H^\bullet_{\text{abs}}(M, E))
\]

**Theorem (Y. Lee)**

Under above assumptions. Then:

\[
\mu \left( \rho_{an,P_-}(\nabla) \otimes \hat{\rho}_{an,P_+}(\nabla) \right) = \pm \rho_{an,\text{rel/abs}}(\nabla) \cdot e^{i\pi \frac{\text{rk}E \cdot \chi(M,C)}}.
\]
Comparison theorem for refined analytic torsions

- \( \hat{\rho}_{\text{an}, \mathcal{P}_+}(\nabla) \) refined analytic torsion defined by \(-\Gamma\) instead of \(\Gamma\).
- The fusion isomorphism

\[
\mu : \det \mathcal{H}^\bullet_{\mathcal{P}_0}(M, E) \otimes \det \mathcal{H}^\bullet_{\mathcal{P}_1}(M, E) \to \det(\mathcal{H}^\bullet_{\text{rel}}(M, E) \oplus \mathcal{H}^\bullet_{\text{abs}}(M, E))
\]

**Theorem (— Y. Lee)**

Under above assumptions. Then:

\[
\mu \left( \rho_{\text{an}, \mathcal{P}_-}(\nabla) \otimes \hat{\rho}_{\text{an}, \mathcal{P}_+}(\nabla) \right) = \pm \rho_{\text{an, rel/abs}}(\nabla) \cdot e^{\frac{i\pi}{2} \text{rk} E \cdot \chi(M, C)}.
\]
Thank you!