On the holonomy fibration

Eckhard Meinrenken
based on work with Alejandro Cabrera and Marco Gualtieri

Workshop on
Geometric Quantization
Adelaide, July 2015
General theme:

Hamiltonian $LG$-spaces $\leftrightarrow$ q-Hamiltonian $G$-spaces

Many aspects of this correspondence are not fully understood.
<table>
<thead>
<tr>
<th></th>
<th>Ham. $LG$-spaces</th>
<th>q-Ham. $G$-spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume forms</td>
<td>?</td>
<td>Y</td>
</tr>
<tr>
<td>Equivariant cohomology</td>
<td>(?)</td>
<td>Y</td>
</tr>
<tr>
<td>Kirwan surjectivity</td>
<td>Y</td>
<td>(Y)</td>
</tr>
<tr>
<td>Norm-square localization</td>
<td>Y</td>
<td>(?)</td>
</tr>
<tr>
<td>Quantization</td>
<td>(?)</td>
<td>Y</td>
</tr>
<tr>
<td>Kähler structures</td>
<td>(Y)</td>
<td>?</td>
</tr>
<tr>
<td>‘Poisson’ structures</td>
<td>?</td>
<td>(Y)</td>
</tr>
</tbody>
</table>

Introduction

Eckhard Meinrenken
On the holonomy fibration
Holonomy fibration

\[ A = \Omega^1(S^1, g) \circ LG \]

\[ g.A = \text{Ad}_g(A) - \partial g g^{-1}. \]

Suppose \( g \) has invariant metric, denoted “·”. \( \rightsquigarrow \) central extension:

\[ \hat{L}_g = \mathbb{R} \oplus L_g \]

with bracket

\[ [t_1 + \xi_1, t_2 + \xi_2]_{\hat{L}_g} = \int_{S^1} \xi_1 \cdot \partial \xi_2 + [\xi_1, \xi_2]_{L_g}. \]

Have \( LG \)-equivariant isomorphism

\[ A = \{1\} \times L_g^* \subset \hat{L}_g^*. \]
Basic facts from Poisson geometry

- Let $\mathfrak{t} = \text{Lie}(K)$. Then $\mathfrak{t}^*$ has a $K$-invariant Poisson structure. (Leaves = coadjoint orbits.)
- For central extension

$$0 \to \mathbb{R} \to \hat{\mathfrak{k}} \to \mathfrak{k} \to 0$$

the level sets $\{\lambda\} \times \mathfrak{t}^* \subset \hat{\mathfrak{k}}^*$ are $K$-equivariant Poisson submanifolds.

- If $M$ is Poisson, $H \circ M$ a principal action by Poisson automorphisms, then $M/H$ is Poisson.
We’d like to apply this to our setting:

\[ 0 \to \mathbb{R} \to \hat{L}_g \to L_g \to 0 \]

with \( L_G \) acting on

\[ A = \{1\} \times L_g^* \subset \hat{L}_g^* \]

(symplectic leaves = coadjoint \( L_G \)-orbits) and

\[ A/L_0 G \cong G. \]

But: \( G \) does not have a reasonable \( L_G/L_0 G \cong G \)-equivariant Poisson structure.

Problem: \( \dim A = \infty, \ \dim L_G = \infty. \)
It’s not even obvious what we mean by ‘Poisson structure’ on $\mathcal{A}$.
- Bivector field $\pi \in \Gamma(\wedge^2 T\mathcal{A})$? Infinite rank??
- Bilinear forms $\{\cdot, \cdot\}$ on smooth functions? Domain??

We’ll show: The ‘Lie-Poisson structure’ on $\mathcal{A}$ makes sense as a Dirac structure, and descends to a Dirac structure on $G$. 
Dirac geometry was introduced by T. Courant and A. Weinstein in 1989.

\[ T^*M = TM \oplus T^*M, \]
\[ \langle v_1 + \alpha_1, v_2 + \alpha_2 \rangle = \alpha_1(v_2) + \alpha_2(v_1), \]
\[ [v_1 + \alpha_1, v_2 + \alpha_2] = [v_1, v_2] + \mathcal{L}_{v_1} \alpha_2 - \iota_{v_2} d\alpha_1 + \iota_{v_1} \iota_{v_2} \eta. \]

where \( \eta \in \Omega^3_{cl}(M) \).

**Definition**

\( E \subset TM \) is a **Dirac structure** if

\( E = E^\perp, \)

\( \Gamma(E) \) closed under \( [\cdot, \cdot] \).
Dirac geometry

Examples

1. $\omega \in \Omega^2(M) \leadsto \text{Graph}(\omega)$ is a Dirac structure $\iff d\omega = 0$.
2. $\pi \in \Gamma(\wedge^2 TM) \leadsto \text{Graph}(\pi)$ is a Dirac structure $\iff \pi$ is Poisson.
3. Conversely, a Dirac structure $E \subseteq TM$ is a Poisson structure $\iff E \cap TM = 0$.
4. Lie-Poisson structure: $E \subseteq Tg^*$ spanned by sections
   \[ e(\xi) = \xi_{g^*} + \langle d\mu, \xi \rangle, \quad \xi \in g. \]
5. Cartan-Dirac structure: $E \subseteq TG$ spanned by sections
   \[ e(\xi) = \xi_G + \frac{1}{2}(\theta^L + \theta^R) \cdot \xi, \quad \xi \in g \]
   is a Dirac structure wrt $\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G)$. Here
   \[ \theta^L = g^{-1} dg, \quad \theta^R = dg \, g^{-1}. \]
The definitions also work for Hilbert manifolds $M$.

**Definition**

A Dirac structure $E \subseteq TM$ is called a

- Poisson structure $\iff E \oplus TM = TM$,
- weak Poisson structure $\iff E \cap TM = 0$.

The leaves of a weak Poisson structure are weakly symplectic.
Example

- $\mathcal{A} = \Omega^1_{H^r}(S^1, g)$ connections of Sobolev class $r \geq 0$
- $LG = \text{Map}_{H^{r+1}}(S^1, G)$ loop group
- Dirac structure $E \subseteq \mathbb{T}\mathcal{A}$ spanned by

$$e(\xi) = \xi_{\mathcal{A}} + \langle d\mathcal{A}, \xi \rangle, \quad \xi \in Lg$$

where

$$\xi_{\mathcal{A}}|_{\mathcal{A}} = \partial \xi + [A, \xi].$$

Then $E$ is a weak Poisson structure: $E \cap T\mathcal{A} = 0$. But $T\mathcal{A} \neq E \oplus T\mathcal{A}$. Note

$$E|_{\mathcal{A}} = \text{graph}(\partial_{\mathcal{A}}), \quad \partial_{\mathcal{A}} : \Omega^0_{H^{r+1}}(S^1, g) \to \Omega^1_{H^r}(S^1, g)$$

skew-adjoint operator $\partial_{\mathcal{A}} = \partial + \text{ad}_A$ with dense domain.
Reduction of Dirac structures

Suppose \( H \circ M \) preserves \( \eta \in \Omega^3_{cl}(M) \). Then \( H \circ T M \) by automorphisms of \( \langle \cdot, \cdot \rangle, \; \lbrack \cdot, \cdot \rbrack \).

**Definition**

\( \rho : \mathfrak{h} \to \Gamma(TM) \) defines generators for the action if

\[
\lbrack \rho(\xi), \cdot \rbrack = \left. \frac{d}{dt} \right|_{t=0} \exp(-t\xi)^* 
\]
on \( \Gamma(TM) \).

**Examples**

\( G \circ Tg^* \), \( G \circ TG \), \( LG \circ TA \) have generators \( \rho(\xi) = e(\xi) \).

\[
e(\xi) = \xi g^* + \langle d\mu, \xi \rangle, \quad \xi \in g
\]
\[
e(\xi) = \xi G + \frac{1}{2}(\theta^L + \theta^R) \cdot \xi, \quad \xi \in g
\]
\[
e(\xi) = \xi A + \langle dA, \xi \rangle, \quad \xi \in Lg
\]
Suppose $H \circ M$ is a principal action preserving $\eta$, and that

$$\rho: M \times \mathfrak{h} \to TM$$

defines generators for $H \circ TM$.

**Theorem (Bursztyn-Cavalcanti-Gualtieri)**

Suppose $J = \rho(M \times \mathfrak{h}) \subseteq TM$ is isotropic. Then $\langle \cdot, \cdot \rangle$, $[\cdot, \cdot]$ descend to

$$TM//H = (J^\perp/J)/H.$$ 

Furthermore, if $E \subseteq TM$ is an $H$-invariant Dirac structure with $J \subseteq E$, then

$$E//H = (E/J)/H$$

is a Dirac structure.

One has $TM//H \cong T(M/H)$ but this depends on a choice.
Reduction of Dirac structures

Have an exact sequence

\[ 0 \to T^*(M/H) \to \mathbb{T}M \bowtie H \to T(M/H) \to 0. \]

To identify \( \mathbb{T}M \bowtie H \cong T(M/H) \) one needs an isotropic splitting

\[ T(M/H) \to (\mathbb{T}M) \bowtie H. \]

Equivalently, need \( H \)-equivariant isotropic splitting \( j : TM \to \mathbb{T}M \) with \( J \subseteq j(TM) \).

Such a splitting is described by a 2-form \( \varpi \), with \( d\varpi = p^* \eta \).
Given $H \triangleleft M$ and $\varrho: M \times \mathfrak{h} \to \mathbb{T}M$ as above, we have:

Any connection 1-form $\theta \in \Omega^1(M, \mathfrak{h})^H$ determines an isotropic splitting.

Explicitly, writing $\varrho(\xi) = \xi_M + \alpha(\xi)$, and letting

$$c(\xi_1, \xi_2) = \iota((\xi_1)_M)\alpha(\xi_2),$$

we have

$$\varpi = -\alpha(\theta) + \frac{1}{2}c(\theta, \theta).$$
Reduction of the Lie-Poisson structure on \( \mathcal{A} \)

Back to our setting:

\[
\begin{align*}
LG \cap E \subseteq TA, & \quad \varrho: Lg \to \Gamma(TA), & \quad \mathcal{A}/L_0G = G.
\end{align*}
\]

\textbf{Hol:} \( \mathcal{A} \to G \) has an ‘almost canonical’ connection \( \theta \in \Omega^1(\mathcal{A}, L_0g) \), depending on choice of a ‘bump function’ on \([0, 1]\). Constructed by \textit{caloron correspondence} of Michael Murray and Raymond Vozzo.

\[
\begin{array}{ccc}
\mathcal{A} & \overset{}{\longrightarrow} & L_0Q \\
\text{/}L_0G & & \text{/}L_0G \\
G & \overset{}{\longrightarrow} & L_0(G \times S^1)
\end{array}
\]

with the \( G \)-bundle \( Q = (G \times \mathbb{R} \times G)/\mathbb{Z} \to G \times S^1 \).
For any choice of bump function, get connection $\theta$ and hence $\varpi \in \Omega^2(\mathcal{A})^{LG}$ as above.

- $\varpi$ independent of choice of bump function.
- $d\varpi = \text{Hol}^* \eta$, with $\eta \in \Omega^3(G)$ the Cartan 3-form.

This 2-form also appeared in the ’98 AMM paper:

$$\varpi = \frac{1}{2} \int_0^1 \text{Hol}_s^* \theta^R \cdot \frac{\partial}{\partial s} \text{Hol}_s^* \theta^R \in \Omega^2(\mathcal{A})$$

where $\text{Hol}_s : \mathcal{A} \to G$ is the holonomy from 0 to $s$. 
Theorem (Cabrera-Gualtieri-M)

The reduction of \((\mathbb{T}A, E)\) by \(L_0 G\), using the standard generators \(\varrho\) and the isotropic splitting defined by \(\varpi\), is the Cartan-Dirac structure on \(G\).
Remark

Can also consider twisted loop groups \(\rightsquigarrow\) twisted Cartan-Dirac structure.

Remark

Similar discussion of \(S^1\) with \(n\) marked points \(\rightsquigarrow\) interesting Dirac structure on \(G^n\). (Cf. Li-Bland, Severa.)
Reduction of morphisms

Let $M_1, M_2$ be manifolds with closed 3-forms $\eta_1, \eta_2$.

A map $\Phi: M_1 \to M_2$ together with a 2-form $\omega \in \Omega^2(M_1)$ defines a relation $(\mathbb{T}M_1)_{\eta_1} \to (\mathbb{T}M_2)_{\eta_2}$, where

$$v_1 + \alpha_1 \sim v_2 + \alpha_2 \iff v_2 = \Phi_* v_1, \quad \alpha_1 = \Phi_* \alpha_2 + \iota v_1 \omega.$$

Given Dirac structures $E_i \subset (\mathbb{T}M_1)_{\eta_i}$, we say that

$$(\Phi, \omega): ((\mathbb{T}M_1)_{\eta_1}, E_1) \to (\mathbb{T}M_2)_{\eta_2}, E_2)$$

is a Dirac morphism if

- $\Phi^* \eta_2 = \eta_1 + d\omega$
- Every $x_2 \in (E_2)_{\Phi(m)}$ is $(\Phi, \omega)$-related to a unique element $x_1 \in (E_1)_m$. 
Example

A Dirac morphism \((\Phi, \omega): (\mathbb{T}M, TM) \rightarrow (\mathbb{T}g^*, E_g^*)\) is a Hamiltonian \(g\)-space.

Example

A Dirac morphism \((\Phi, \omega): (\mathbb{T}M, TM) \rightarrow (\mathbb{T}G, E_G)\) is a quasi-Hamiltonian \(g\)-space.

Example

A Dirac morphism \((\Phi, \omega): (\mathbb{T}M, TM) \rightarrow (\mathbb{T}A, E_A)\) is a Hamiltonian \(Lg\)-space.
Reduction of morphisms

Our results on reductions of Dirac structures extend to morphisms. ⇒ recover the correspondence

Hamiltonian $LG$-spaces $\leftrightarrow$ q-Hamiltonian $G$-spaces
Further directions, open questions

- How to explain the quasi-Poisson structure on $G$ by reduction?
- How to explain the volume forms on q-Hamiltonian spaces by reduction?
- Metric and Kähler aspects?

Many of the foundations of Poisson/Dirac geometry and Lie algebroids in infinite dimensions remain to be developed. (E.g., forthcoming work with Bursztyn and Lima.)