K-theory in Condensed Matter Physics

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References

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Topological phases

In contrast to usual phases, which are related to a spontaneously broken symmetry, topological phases (e.g. topological insulators) are many fermion systems possessing an unusual band structure that leads to a bulk band gap as well as topologically protected gapless extended surface modes.

Topological phases of free fermion models arise from symmetries of one-particle Hamiltonians (time reversal, particle-hole). There are 10 symmetry classes of Hamiltonians (the 'ten-fold way') and non trivial topological phases are classified by K-theory.

Altland-Zirnbauer classes and The Periodic Table

AZ label	TRS	PHS	SLS	$d=0$	$d=1$	$d=2$	$d=3$
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	$+1$	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
D	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	$+1$	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	0	$\mathbb{Z}$	0
CI	$+1$	-1	1	0	0	0	$\mathbb{Z}$
AI	$+1$	0	0	$\mathbb{Z}$	0	0	0

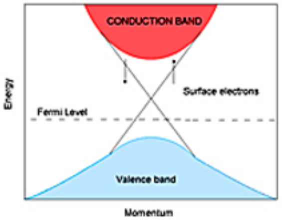
Topological phases, cont'd

The relation to K-theory arises in three different ways:

- Through vector bundles
- Through classifying spaces
- Through extensions of Clifford modules

These are related through the Atiyah-Bott-Shapiro construction.

Topological phases, cont'd



In the presence of translation symmetry, we can block diagonalise the Hamiltonian in terms of eigenvalues under the translation operators

$$H = \bigoplus_{\mathbf{k}\in \mathrm{BZ}} H(\mathbf{k})$$

where $H(\mathbf{k})$ is so-called Bloch Hamiltonian, and BZ is the Brillouin zone (e.g. a torus $\mathbb{T}^d$).

Bands can have nontrivial structure protected under (gap-preserving) deformations of Hamiltonians. I.e. we need to classify deformation classes of Hamiltonians.
It suffices to put the gap at $E = E_F = 0$ and to study 'flattened Hamiltonians', i.e. with eigenvalues ± 1.

Flattened Hamiltonians

If we have an arbitrary gapped Hamiltonian H (with a gap at 0), let $P_{\pm}$ be the projection operator on the positive/negative eigenspace. The flattened Hamiltonian $\widetilde{H}$, with eigenvalues ± 1, is defined as

$$\widetilde{H} = P_+ - P_- = 1 - 2P_- \,.$$

To show that H and $\tilde{H}$ are homotopic, let P_λ be the projection operator onto the eigenspace of eigenvalue λ. We have

$$P_+ = \bigoplus_{\lambda>0} P_\lambda \,, \qquad P_- = \bigoplus_{\lambda<0} P_\lambda$$

Now consider

$$H_t = \bigoplus_{\lambda} \left(\frac{\lambda}{(1-t) + t|\lambda|} \right) P_\lambda \,, \qquad t \in [0,1] \,.$$

Then

$$H_0 = \bigoplus_{\lambda} \lambda P_\lambda = H \,, \qquad H_1 = \bigoplus_{\lambda} \frac{\lambda}{|\lambda|} P_\lambda = \bigoplus_{\lambda>0} P_\lambda - \bigoplus_{\lambda<0} P_\lambda = \widetilde{H} \,.$$

The Example

Consider $\mathcal{H} = \mathbb{C}^2$. In terms of Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} , \quad (\sigma_i)^\dagger = \sigma_i$$

we can define a Hamiltonian

$$H = H(\widehat{\mathbf{x}}) = \sum_i \widehat{x}^i \sigma_i = \widehat{\mathbf{x}} \cdot \sigma = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

with $\widehat{\mathbf{x}} = (x, y, z) \in S^2$.

The Example, cont'd

We have

$$\text{Tr } H = 0 \,, \qquad H^\dagger = H \,, \qquad H^2 = 1 \,,$$

from which we conclude that H has eigenvalues ± 1, each with multiplicity 1.

For eigenvalue $\lambda = -1$ (the 'valence band') the normalised eigenvectors $\psi_-^{N/S}$ on $S^2_{N/S}$, where $S^2_N = S^2 \backslash \{z = -1\}$ and $S^2_S = S^2 \backslash \{z = 1\}$, are given by

$$\psi_-^N = \frac{1}{\sqrt{2(1+z)}} \begin{pmatrix} x - iy \\ -(1+z) \end{pmatrix} \,, \qquad \psi_-^S = \frac{1}{\sqrt{2(1-z)}} \begin{pmatrix} -(1-z) \\ x + iy \end{pmatrix}$$

Together they define a linebundle E_- over S^2, with first Chern class $c_1 = 1$. [Associated circle bundle is the Hopf fibration.]

The Example, cont'd

Knowing the eigenbundle E_-, we can reconstruct the Hamiltonian as follows. First we determine the projection operator $P_- : E \to E_-$, where E is the trivial $\mathbb{C}^2$-bundle over S^2

$$P_- = \psi_-^N \psi_-^{N\dagger} = \frac{1}{2}\begin{pmatrix} 1-z & -(x-iy) \\ -(x+iy) & 1+z \end{pmatrix},$$

and hence

$$H = P_+ - P_- = 1 - 2P_-$$

The Example, cont'd

A connection A_- on E_- is given, locally on $S^2_{N/S}$, by

$$A^N_- = i\psi^{N\dagger}_- d\psi^N_- = \frac{xdy - ydx}{2(1+z)} = \frac{\sin^2\theta \; d\phi}{2(1+\cos\theta)} = \tfrac{1}{2}(1-\cos\theta)\; d\phi$$

$$A^S_- = i\psi^{S\dagger}_- d\psi^S_- = \frac{-xdy + ydx}{2(1-z)} = \frac{-\sin^2\theta \; d\phi}{2(1-\cos\theta)} = -\tfrac{1}{2}(1+\cos\theta)\; d\phi$$

which is precisely the connection for a Dirac monopole.
On $S^2_N \cap S^2_S$ the $A^{N/S}_-$ differ by a gauge transformation

$$A^N_- - A^S_- = d\phi\,.$$

Thus

$$F_- = dA^N_- = dA^S_- = \tfrac{1}{2}\sin\theta \; d\theta \wedge d\phi\,,$$

is globally defined on S^2, and

$$c_1 = \frac{1}{2\pi}\int_{S^2} F_- = \frac{1}{4\pi}\mathrm{Vol}(S^2) = 1\,.$$

Let Ψ be an $N \times k$ matrix of k (orthonormal) vectors in $\mathbb{C}^N$. In terms of matrix components Ψ_{Aa}, $A = 1, \ldots, N$, $a = 1, \ldots, k$. We have

$$\Psi^\dagger \Psi = 1 \,.$$

The projections operator P onto the subspace spanned by the vectors Ψ_a, is given by

$$P = \Psi \Psi^\dagger \,, \qquad P^2 = P \,.$$

Now consider a smooth family of projection operators $P = P(\widehat{\mathbf{x}})$, varying over a space X, and the subbundle $E \subset X \times \mathbb{C}^N$ given by P.

Projection operators and Berry connections, cont'd

On E, we can canonically construct two connections ∇

- $\nabla s = Pd(Ps) = Pds + (PdP)s$.

$$\begin{aligned}\nabla^2 s &= Pd(Pds + PdPs) + PdP \wedge (Pds + PdPs) \\ &= PdP \wedge ds + PdP \wedge dPs - PdP \wedge ds + PdPP \wedge ds \\ &\quad + PdP \wedge PdPs = (PdP \wedge dP)s \equiv F_\nabla s\,,\end{aligned}$$

  i.e. , curvature $F_\nabla = P\,dP \wedge dP$.

- $Ds = \Psi^\dagger d(\Psi s) = ds + (\Psi^\dagger d\Psi)s$, with curvature $F_D = d\Psi^\dagger \wedge d\Psi + \Psi^\dagger d\Psi \wedge \Psi^\dagger d\Psi$ (Berry connection).

They are related by

$$F_\nabla = \Psi F_D \Psi^\dagger$$

Projection operators and Berry connections, cont'd

Proof: From $P = \Psi\Psi^\dagger$, and $\Psi^\dagger\Psi = 1$, it follows

$$dP = d\Psi\Psi^\dagger + \Psi d\Psi^\dagger\,, \qquad d\Psi^\dagger\Psi + \Psi^\dagger d\Psi = 0$$

From $P^2 = P$ it follows

$$PdP + dPP = dP$$

Multiplying by P on the left (or right), then gives $PdPP = 0$. Differentiating this equation gives $PdP \wedge dP = dP \wedge dPP$. Hence

$$\begin{aligned} F_\nabla &= PdP \wedge dP = P^2 dP \wedge dP = PdP \wedge dPP \\ &= \Psi\Psi^\dagger(d\Psi\Psi^\dagger + \Psi d\Psi^\dagger) \wedge (d\Psi\Psi^\dagger + \Psi d\Psi^\dagger)\Psi\Psi^\dagger \\ &= \Psi(d\Psi^\dagger \wedge d\Psi + \Psi^\dagger d\Psi \wedge \Psi^\dagger d\Psi)\Psi^\dagger \\ &= \Psi F_D \Psi^\dagger\,, \end{aligned}$$

Projection operators and Berry connections, cont'd

In particular we find

$$\mathrm{Tr}({F_\nabla}^n) = \mathrm{Tr}(P(dP)^{2n}) = \mathrm{tr}({F_D}^n)\,.$$

where Tr is taken over $\mathbb{C}^N$ and tr over $\mathbb{C}^k$.

In particular, for $P = \frac{1}{2}(1 - H)$,

$$c_1 = \frac{i}{2\pi} \int \mathrm{Tr}(P\, dP \wedge dP) = -\frac{i}{16\pi} \int \mathrm{Tr}(H\, dH \wedge dH)$$

The Example, cont'd

E.g. for $H = \widehat{\mathbf{x}} \cdot \sigma$, we have

$$\begin{aligned} c_1 &= -\frac{i}{16\pi} \int \mathrm{Tr}(H\, dH \wedge dH) \\ &= -\frac{i}{16\pi} \int d^2x\ \epsilon^{\mu\nu}\ \widehat{x}^i \partial_\mu \widehat{x}^j \partial_\nu \widehat{x}^k \mathrm{Tr}(\sigma_i \sigma_j \sigma_k) \\ &= \frac{1}{8\pi} \int d^2x\ \epsilon^{\mu\nu}\ \widehat{\mathbf{x}} \cdot (\partial_\mu \widehat{\mathbf{x}} \times \partial_\nu \widehat{\mathbf{x}}) = 1 \end{aligned}$$

Consider the generalization

$$H = \widehat{h}(\mathbf{x}) \cdot \sigma\,, \qquad \widehat{h} : S^2 \to S^2$$

gives negative eigenvector bundle with

$$c_1 = \frac{1}{8\pi} \int_{S^2} d^2x\ \epsilon^{\mu\nu} \widehat{h} \cdot (\partial_\mu \widehat{h} \times \partial_\nu \widehat{h})$$

i.e. winding number of $\widehat{h}$, e.g. element of $\pi_2(S^2) \cong \mathbb{Z}$.

The Example, further generalizations

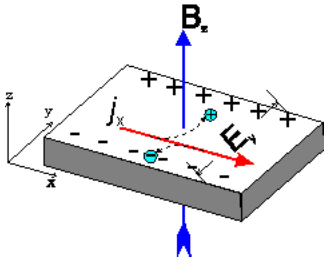
Instead, in the previous example, we can take $\widehat{h} : X \to S^2$, or more generally $h : X \to S^d$ if we have a higher dimensional generalization of the Pauli matrices (representation of a Clifford algebra)

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

I.e.

$$H = \widehat{h}(\mathbf{x}) \cdot \gamma$$

Integer Quantum Hall Effect



The Kubo formula for the Hall conductance σ_{xy}

$$j_x = \sigma_{xy} E_y$$

gives

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} n$$

where

$$n = c_1 = \frac{1}{2\pi} \int_{BZ} \mathrm{tr}\, F_D$$

Integer Quantum Hall Effect, cont'd

Determine deformation classes of Hamiltonians only up to addition of trivial valence bands (physical properties are the same). I.e. to the negative eigenbundle E_- we associate its class in $K^0(X)$.

Classifying spaces

We may parametrize our Hamiltonian as

$$H = A(\mathbf{x})\sigma_z A(\mathbf{x})^\dagger$$

where $A : X \to U(2)$. In fact, since $U(1) \times U(1) \subset U(2)$ commutes with σ_z, we have

$$A : X \to U(2)/U(1) \times U(1) \cong S^2 \,.$$

For $N \to \infty$, the symmetric space $\oplus_k U(N)/U(k) \times U(N-k)$ approaches the classifying space C_0,

$$\mathrm{K}^0(X) = [X, C_0]$$

In particular $[pt, C_0] \cong \pi_0(C_0) \cong \mathbb{Z}$, $[S^2, C_0] \cong \pi_2(C_0) \cong \mathbb{Z}$.

Symmetries

In a QM system we are interested in transformations $A : \mathcal{H} \to \mathcal{H}$ such that

$$|\langle Ax, Ay\rangle| = |\langle x, y\rangle| \qquad (*)$$

Theorem (Wigner, 1931)

A surjective map A, satisfying $(\star)$, is of the form $A = cU$, where $|c| = 1$ and U is either a unitary or anti-unitary transformation

Definition

An anti-unitary transformation $U : \mathcal{H} \to \mathcal{H}$ is an anti-linear transformation

$$U(\lambda x + \mu y) = \bar{\lambda} U(x) + \bar{\mu} U(y)$$

such that

$$\langle Ux, Uy\rangle = \overline{\langle x, y\rangle}$$

Symmetries, cont'd

Examples of anti-unitary transformations

- $K : \mathbb{C} \to \mathbb{C}, \quad Kz = \bar{z}, \quad K^2 = 1$
- $U = \sigma_y K = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} K : \mathbb{C}^2 \to \mathbb{C}^2, \quad U^2 = -1$

In QM systems there are three relevant symmetries

- Time Reversal Symmetry (TRS):
  $TH = HT$, $T^2 = \pm 1$ (anti-unitary)
- Particle-Hole Symmetry (PHS) (Charge Conjugation):
  $PH = -HP$, $P^2 = \pm 1$ (anti-unitary)
- Sublattice Symmetry (SLS) (Chiral):
  $CH = -HC$, $C = PT$, $C^2 = 1$ (unitary)

Symmetries, cont'd

In case of Bloch Hamiltonians

- Time Reversal Symmetry (TRS):
  $TH(\mathbf{k}) = H(-\mathbf{k})T$, $T^2 = \pm 1$ (anti-unitary)
- Particle-Hole Symmetry (PHS) (Charge Conjugation):
  $PH(\mathbf{k}) = -H(-\mathbf{k})P$, $P^2 = \pm 1$ (anti-unitary)
- Sublattice Symmetry (SLS) (Chiral):
  $CH(\mathbf{k}) = -H(\mathbf{k})C$, $C = PT$, $C^2 = 1$ (unitary)

There are 3×3 possible choices for T^2, P^2, denoted as $0, \pm 1$, and for $T = P = 0$, there are two choices for C, denoted as $0, 1$.

This leads to 10 symmetry classes [Dyson, Altand-Zirnbauer]

Altland-Zirnbauer classes and The Periodic Table

AZ label	TRS	PHS	SLS	$d=0$	$d=1$	$d=2$	$d=3$
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	$+1$	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
D	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	$+1$	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	0	$\mathbb{Z}$	0
CI	$+1$	-1	1	0	0	0	$\mathbb{Z}$
AI	$+1$	0	0	$\mathbb{Z}$	0	0	0

Classifying spaces

AZ label	Class. Space	G/H	π_0
A	C_0	$\oplus_k(U(N)/U(N-k)\times U(k))$	$\mathbb{Z}$
AIII	C_1	$U(N)\times U(N)/U(N)$	0
BDI	R_1	$O(N)\times O(N)/O(N)$	$\mathbb{Z}_2$
D	R_2	$O(2N)/U(N)$	$\mathbb{Z}_2$
DIII	R_3	$U(2N)/Sp(N)$	0
AII	R_4	$\oplus_k(Sp(N)/Sp(N-k)\times Sp(k))$	$\mathbb{Z}$
CII	R_5	$Sp(N)\times Sp(N)/Sp(N)$	0
C	R_6	$Sp(N)/U(N)$	0
CI	R_7	$U(N)/O(N)$	0
AI	R_0	$\oplus_k(O(N)/O(N-k)\times O(k))$	$\mathbb{Z}$

Another example, free fermion systems

Free fermion Dirac operators

$$\{a_j^\dagger, a_k\} = \delta_{jk}\,, \qquad j,k = 1,\ldots,n$$

A general Hamiltonian conserving particle number is of the form

$$H_A = \sum_{i,j} A_{jk} a_j^\dagger a_k\,, \qquad A^\dagger = A$$

If particle number is not conserved, introduce Majorana operators

$$c_{2j-1} = a_j^\dagger + a_j\,, \qquad c_{2j} = i(a_j^\dagger - a_j)$$

satisfying

$$\{c_l, c_m\} = 2\delta_{lm}\,, \quad l,m = 1,\ldots,2n\,, \qquad c_l^\dagger = c_l$$

Another example, free fermion systems

Free field Hamiltonian (Majorana chain)

$$H_A = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k \, ,$$

where A is real, skew-symmetric, of size $2n$.
Trivial Hamiltonian

$$H_{\text{triv}} = \sum_j (a_j^\dagger a_j - \tfrac{1}{2}) = H_Q$$

where

$$Q = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Another example, free fermion systems

After spectral flattening $H_A \to \widetilde{H_A} = H_{\widetilde{A}}$, we have

$$\widetilde{A} = SQS^{-1}\,, \qquad S \in O(2n)$$

The set of matrices in $O(2n)$ commuting with Q form a subgroup $U(n) \subset O(2n)$, hence $\widetilde{A}$ takes values in $O(2n)/U(n)$. Upon identifying $\widetilde{A} \sim \widetilde{A} \oplus Q$ we find

$$[\widetilde{A}] \in R_2 = \lim_{n\to\infty} O(2n)/U(n)$$

Connected components $\pi_0(R_2) \cong \mathbb{Z}_2$ distinguished by value of $\mathrm{sgn}(\mathrm{Pf}(A)) = \mathrm{Pf}(\widetilde{A}) = \det S = \pm 1$ (particle number mod 2).

Real K-theory

It turns out that the R_q are the classifying spaces for Atiyah's real K-theory, in particular

$$\widetilde{\mathrm{KO}}^{-q}(\mathrm{pt}) \cong \pi_0(R_q)$$

Generalization to higher dimensional parameter spaces X is a little more subtle.

Clifford algebras

$\mathrm{Cl}_{p,q}$ is the algebra (over $\mathbb{R}$) generated by $e_i, i = 1, \ldots, p+q$, with

$$\begin{aligned} e_i^2 &= -1 & i &= 1, \ldots, p \\ e_i^2 &= 1 & i &= p+1, \ldots, p+q \\ e_i e_j + e_j e_i &= 0 & i &\neq j \end{aligned}$$

We have the folllowing isomorphisms

$$\begin{aligned} \mathrm{Cl}_{p,0} \otimes \mathrm{Cl}_{0,2} &\cong \mathrm{Cl}_{0,p+2} \\ \mathrm{Cl}_{0,p} \otimes \mathrm{Cl}_{2,0} &\cong \mathrm{Cl}_{p+2,0} \\ \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1} &\cong \mathrm{Cl}_{p+1,q+1} \\ \mathrm{Cl}_{p+8,0} &\cong \mathrm{Cl}_{p,0} \otimes \mathbb{R}(16) \end{aligned}$$

For Clifford algebras over $\mathbb{C}$ we have $\mathrm{Cl}_{p+2} \cong \mathrm{Cl}_p \otimes \mathbb{C}(2)$.

Clifford algebras

We have the following result for the first few Clifford algebras

$$\mathrm{Cl}_{1,0} \cong \mathbb{C} \qquad \mathrm{Cl}_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$$
$$\mathrm{Cl}_{2,0} \cong \mathbb{H} \qquad \mathrm{Cl}_{0,2} \cong \mathbb{R}(2)$$
$$\mathrm{Cl}_{1,1} \cong \mathbb{R}(2)$$

Classification of Clifford algebras

k	$\mathrm{Cl}_{k,0}(\mathbb{R})$	$\mathrm{Cl}_{0,k}(\mathbb{R})$	$\mathrm{Cl}_k(\mathbb{C})$
0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$
1	$\mathbb{C}(1)$	$\mathbb{R}(1) \oplus \mathbb{R}(1)$	$\mathbb{C}(1) \oplus \mathbb{C}(1)$
2	$\mathbb{H}(1)$	$\mathbb{R}(2)$	$\mathbb{C}(2)$
3	$\mathbb{H}(1) \oplus \mathbb{H}(1)$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{C}(4)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$

Table: Clifford algebras $\mathrm{Cl}_{k,0}$, $\mathrm{Cl}_{0,k}$ and Cl_k

Classification of real Clifford algebras

$p \backslash q$	0	1	2	3	4	5	6	7	8
0	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}^2(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
1	$\mathbb{C}$	$\mathbb{R}(2)$	$\mathbb{R}^2(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}^2(4)$	$\mathbb{H}(8)$	
2	$\mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}^2(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$		
3	$\mathbb{H}^2$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}^2(8)$	$\mathbb{R}(16)$			
4	$\mathbb{H}(2)$	$\mathbb{H}^2(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$				
5	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}^2(4)$	$\mathbb{H}(8)$					
6	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$						
7	$\mathbb{R}^2(8)$	$\mathbb{R}(16)$							
8	$\mathbb{R}(16)$								

Table: Clifford algebras $\mathrm{Cl}_{p,q}$

Clifford modules

Define $N(\mathrm{Cl}_{p,q})$ to be the Grothendieck group of real modules of $\mathrm{Cl}_{p,q}$, i.e. the additive free group generated by the irreducible real modules of $\mathrm{Cl}_{p,q}$.

Let $\imath : \mathrm{Cl}_{p,q} \to \mathrm{Cl}_{p+1,q}$ denote the obvious inclusions of Clifford algebras.

They give rise to the following maps on the Grothendieck groups of modules

$$\imath^* : N(\mathrm{Cl}_{p+1,q}) \to N(\mathrm{Cl}_{p,q})$$

Let us denote $A_{p,q} = N(\mathrm{Cl}_{p,q})/\imath^* N(\mathrm{Cl}_{p+1,q})$

Clifford modules,cont'd

k	$\mathrm{Cl}_{k,0}$	d_k	$N(\mathrm{Cl}_{k,0})$	A_k
0	$\mathbb{R}$	1	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$\mathbb{C}$	2	$\mathbb{Z}$	$\mathbb{Z}_2$
2	$\mathbb{H}(1)$	4	$\mathbb{Z}$	0
3	$\mathbb{H} \oplus \mathbb{H}$	4	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$
4	$\mathbb{H}(2)$	8	$\mathbb{Z}$	0
5	$\mathbb{C}(4)$	8	$\mathbb{Z}$	0
6	$\mathbb{R}(8)$	8	$\mathbb{Z}$	0
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	8	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$
8	$\mathbb{R}(16)$	16	$\mathbb{Z}$	$\mathbb{Z}_2$

Table: Extensions of Clifford modules

Clifford modules, cont'd

$p\backslash q$	0	1	2	3	4	5	6	7
0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
2	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
3	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
4	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
5	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
6	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
7	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Table: Table of $A_{p,q}$

Extension of Clifford Modules; Classifying spaces

Suppose we have a representation of $\mathrm{Cl}_{k,0}$ in $O(16r)$

$$J_i J_j + J_j J_i = -2\delta_{ij}$$

Let G_1 be the subgroup of $O(16r)$ that commutes with J_1, G_2 the subgroup of G_1 that commutes with J_2, etc. We get the following chain of subgroups

$$\begin{aligned} O(16r) \underset{R_2}{\supset} U(8r) \underset{R_3}{\supset} Sp(4r) \underset{R_4}{\supset} Sp(2r) \times Sp(2r) \underset{R_5}{\supset} Sp(2r) \\ \underset{R_6}{\supset} U(2r) \underset{R_7}{\supset} O(2r) \underset{R_0}{\supset} O(r) \times O(r) \underset{R_1}{\supset} O(r) \supset \ldots \end{aligned}$$

Subsequent quotients parametrize the extensions of $\mathrm{Cl}_{p,0}$ to $\mathrm{Cl}_{p+1,0}$. These are precisely the symmetric spaces (classifying spaces) encountered before.

THANKS