Twisted K-theory constructions

Jouko Mickelsson

Department of Mathematics and Statistics
University of Helsinki

April 5, 2013 / Adelaide

Decomposable gerbe class

Twisted K-theory on a manifold X, with twisting in the 3rd integral cohomology, is discussed in the case when X is a product of a circle $\mathbb{T}$ and a manifold M. The twist is assumed to be decomposable as a cup product of the basic integral one form on $\mathbb{T}$ and an integral class in $H^2(M, \mathbb{Z})$. This case was studied some time ago by V. Mathai, R. Melrose, and I.M. Singer. Our aim is to give an explicit construction for the twisted K-theory classes using a quantum field theory model, in the same spirit as the supersymmetric Wess-Zumino-Witten model is used for constructing (equivariant) twisted K-theory classes on compact Lie groups.

K-theory on a topological space X can be twisted by an integral cohomology class σ of degree 3. The nontorsion case involves intrinsically infinite dimensional geometry since the class σ is the characteristic class of a principal bundle with the structure group $PU(H)$, the projective unitary group of an infinite dimensional separable complex Hilbert space H. Partly because of this reason concrete constructions are available only in few cases. Best known of these is twisted K-theory on a compact Lie group G. It was shown by Freed, Hopkins, and Teleman that in the G equivariant case the K-theory $K^*(G, \sigma)$ has a ring structure isomorphic to the Verlinde ring in conformal field theory. Concretely, the twisted -theory classes can be constructed from the quantized supersymmetric Wess-Zumino-Witten model.

This talk is based on a paper by Antti Harju and J.M, arXiv:1208.4921

We shall concentrate on the case $X = \mathbb{T} \times M$, where M is a compact manifold, $\mathbb{T} = \mathbb{T}_\phi = S^1$ is a unit circle and the class σ is represented as a product $\sigma = \frac{d\phi}{2\pi} \wedge \frac{\beta_M}{2\pi i}$ of the 1-form on $\mathbb{T}_\phi$ and a closed integral 2-form on M. This case was already studied by Mathai, Melrose, and Singer. In particular, a formula for the Chern character was derived in the decomposable case. The Chern character does not directly see the torsion classes in $K^*(X, \sigma)$. For this reason we want to analyze closer the torsion classes. We also give a concrete formula for representatives of those classes using a quantum field theory construction similar to [J.Mickelsson, 2002] in the case of a compact simply connected Lie group. As a particular case, we have a construction for the (nonequivariant) torsion classes when M is a torus.

The Dixmier-Douady class from the families Index Theorem

The Hamiltonian quantization of fermionic fields produce a projective bundle of Fock spaces over the parameter space of the Dirac family. The projective bundle defines a gerbe which is topologically characterized by a Dixmier-Douady 3-cohomology class. Especially, we can lift the projective Fock bundle to a vector bundle if and only if the Dixmier-Douady class is zero.
The de Rham representative of the Dixmier-Douady class is the 3-form part of the local index theory of the Dirac family.
We consider a manifold of type $\mathbb{T} \times M$ with a nontrivial decomposable integral 3-cohomology class,

$$\sigma = \frac{d\phi}{2\pi} \wedge \frac{\beta_M}{2\pi i},$$

where $\beta_M \in H^2(M, 2\pi i\mathbb{Z})$. We are interested in K-theory twisted by a gerbe and therefore we can exploit the Hamiltonian quantization to build a gerbe over $\mathbb{T}_\phi \times M$.

The first goal is to construct a family of Dirac operators on $\mathbb{T}_\phi \times M$ with a three form component in its index given by the decomposable class σ.
Consider a 2-torus $\mathbb{T}^2$ with angle variables (θ, ϕ). We choose an open cover $\{\mathbb{T}_+, \mathbb{T}_-\}$ for $\mathbb{T}$ such that $\mathbb{T}_{+-} = \mathbb{T}_+ \cap \mathbb{T}_-$ consists of two disconnected arcs, one which is a neighbourhood of -1 and another a neighbourhood of 1. We denote these by $\mathbb{T}_{+-}^{(-1)}$ and $\mathbb{T}_{+-}^{(1)}$.
The isomorphism classes of line bundles over $\mathbb{T}^2$ are classified by $\mathbb{Z}$ since $H^2(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z}$. The bundle λ_1 corresponding to a generator of the cohomology group can be described as follows: if ψ is a smooth section of λ_1, then $\psi(\theta, \phi + 2\pi) = e^{i\theta}\psi(\theta, \phi)$.

After puling back with the map $\mathbf{R} \times \mathbf{T} \to \mathbf{T} \times \mathbf{T}$, sending ϕ to ϕ mod 2π, a connection of this bundle can be defined by

$$\nabla_1 = d\theta \otimes \partial_\theta + d\phi \otimes \partial_\phi - \frac{i}{2\pi} d\theta \otimes \phi.$$

The curvature of the connection is the cocycle in de Rham cohomology

$$\nabla_1^2 = \frac{i}{2\pi} d\theta \wedge d\phi \in H^2(\mathbb{T}^2, 2\pi i \mathbf{Z}).$$

Consider a smooth manifold M with nontrivial second cohomology and fix a line bundle λ with a connection and so that the curvature is equal to $\beta_M \in H^2(M)$ which we assume to be nontrivial. Now $\tilde{\lambda} = \lambda_1 \boxtimes \lambda$ defines a line bundle over $\mathbb{T}^2 \times M$. Consider a smooth fibration

$$\mathbb{T}_\theta \hookrightarrow \mathbb{T}_\theta \times \mathbb{T}_\phi \times M \twoheadrightarrow \mathbb{T}_\phi \times M.$$

At each $(\phi, x) \in \mathbb{T}_\phi \times M$, the bundle $\tilde{\lambda}$ restricted to the fibre $\mathbb{T}_\theta$ defines a line bundle $\lambda(\phi, x) \twoheadrightarrow \mathbb{T}_\theta$. In fact, the sections of this bundle are periodic in the direction θ and therefore at fixed (ϕ, x) the bundle $\lambda(\phi, x)$ is the product $\mathbb{T}_\theta \times \mathbb{C}$.

At each point (ϕ, x) we define a Hilbert space $\mathcal{H}(\phi, x) = L^2(\mathbb{T}_\theta, \lambda(\phi, x))$ of L^2-functions on $\mathbb{T}_\theta$ with values in the fibre $\lambda(\phi, x)$. Then

$$\mathbf{H} = \coprod_{(\phi,x)\in\mathbb{T}\times M} \mathcal{H}(\phi, x)$$

is a locally trivial bundle of Hilbert spaces over $\mathbb{T} \times M$. In fact, it is the trivial bundle with fibre $L^2(\mathbb{T}_\theta, \mathbb{C})$ twisted by the line bundle λ. As a Hilbert bundle it is trivial by Kuiper's theorem. However, considered as a reduced bundle with the structure group of smooth $\mathbb{T}$ valued gauge transformations, the group $L\mathbb{T}_\theta$ of smooth endomorphism of $\mathbb{T}_\theta$, it is nontrivial. The gauge group acts on each fibre $\mathcal{H}(\phi, x)$ by multiplication: $m : L\mathbb{T}_\theta \times \mathcal{H}(\phi, x) \to \mathcal{H}(\phi, x)$. The group $\mathbb{Z}$ of translations over $\mathbb{T}_\phi$ acts on the sections of $\mathcal{H}$ by the rule

$$a.\varphi(\phi, x) = m(e^{ia\theta})\varphi(\phi, x).$$

The free Dirac operator $-i\partial_\theta$ is an unbounded self adjoint operator on each fibre $\mathcal{H}(\phi, x)$. The space of vector potentials on each fibre is given by $\mathcal{A} \simeq C^\infty(\mathbb{T}_\theta) \otimes i\mathbb{R}$. The gauge group $L\mathbb{T}_\theta$ acts on the Dirac operators by conjugation, leading to the action $A \mapsto A + g^{-1}dg$ on gauge potentials. The gauge orbit space is $\mathcal{A}/L\mathbb{T}_\theta$ which can be identified with a circle. Thus, $\mathbb{T}_\phi$ has a natural interpretation of a space of gauge potentials which we twist with the bundle λ on M. Actually, it is sufficient to consider constant vector potentials ϕ parametrized by the real line $\mathbf{R}$. The gauge transformations by $\mathbb{T}$ valued functions $e^{i\theta}$ on $\mathbb{T}_\theta$ change the parameter $\phi \mapsto \phi + 2\pi$, so again the family $-i\partial_\theta + \frac{\phi}{2\pi}$ modulo gauge transformations is parametrized by $\mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$. After twisting this family by the line bundle λ over M we get a family parametrized by $X = \mathbb{T} \times M$. The Dirac family is twisted by the complex line bundle over $\mathbf{T}^2 \times M$ with connection $\nabla_1 \otimes \nabla_M$ and the total curvature

$$F = \frac{i}{2\pi} d\theta \wedge d\phi + \beta_M \in H^2(\mathbb{T}_\theta \times X, 2\pi i\mathbf{Z}).$$

The Dirac family D defines an eigenvalue problem at each $(\phi, x) \in \mathbb{T} \times M$. If we let the angle ϕ vary from 0 to 2π, then there is a translation in the set of eigenvalues as they all grow by 2π. Because of the spectral flow the group element of $K^1(\mathbb{T} \times M)$ defined by the Fredholm family is nontrivial. In fact, the spectral flow produces a nontrivial cocycle of $H^1(\mathbb{T} \times M, \mathbb{Z})$ via the index map. The twisting bundle λ produces another nontrivial class, a three form in $H^{\mathrm{odd}}(\mathbb{T} \times M, \mathbb{Z})$.

The local index formula, $\mathrm{ind} : K^1(\mathbb{T} \times M) \to H^{\mathrm{odd}}(\mathbb{T} \times M)$, gives

$$\begin{aligned}
\mathrm{ind}(D) &= \int_{\mathbb{T}} ch(\lambda_1 \boxtimes \lambda) \\
&= \int_{\mathbb{T}} \exp(\frac{\nabla_1^2}{2\pi i}) \wedge \exp(\frac{\beta_M}{2\pi i}) \\
&= \int_{\mathbb{T}} \exp\left(\frac{1}{4\pi^2} d\theta \wedge d\phi + \frac{\beta_M}{2\pi i}\right) \\
&= \frac{d\phi}{2\pi} + \frac{d\phi}{2\pi} \wedge \frac{\beta_M}{2\pi i} + \cdots .
\end{aligned}$$

The A-roof genus on $\mathbb{T}^2 \times M$ does not contribute on this level in the character formula. The three cohomology part is exactly the decomposable 3-cohomology class.

Let $\mathcal{H}$ be a separable Hilbert space. The algebra A is called a canonical anticommutation relations (CAR) algebra over $\mathcal{H}$ if there is an antilinear mapping $\mathcal{H} \to A$ such that $a(f) : f \in \mathcal{H}$ generate a unital C^*-algebra A which fulfills

$$\{a(u), a(v)\} = 0 \text{ and } \{a(u), a(v)^*\} = \langle u, v\rangle 1$$

for all $u, v \in \mathcal{H}$. The CAR algebra is unique up to C^*-algebra isomorphism.

For a fixed $(\phi, x) \in \mathbb{T} \times M$, the Dirac operator $D_{\phi,x}$ defines a polarization $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ such that $\mathcal{H}^+$ is spanned by the nonnegative eigenstates. A Fock space $\mathcal{F}$ is a Hilbert space with a vacuum vector $|0\rangle$ and the CAR algebra acts on the vacuum such that

$$a(u)|0\rangle = 0 = a^*(v)|0\rangle \text{ for all } u \in \mathcal{H}^+, v \in \mathcal{H}^-,$$

and the basis of a Fock space is spanned by

$$a(u_{i_1}) \cdots a(u_{i_k}) a^*(u_{j_1}) \cdots a^*(u_{j_l})|0\rangle, \text{for} u_{i_\nu} \in \mathcal{H}^-, u_{j_\nu} \in \mathcal{H}^+.$$

We can think of the vacuum as the formal infinite wedge product

$$|0\rangle = u_{-1} \wedge u_{-2} \wedge u_{-3} \wedge \cdots$$

and the general basis vector as

$$u_{j_1} \wedge u_{j_2} \wedge u_{j_3} \wedge \cdots$$

where $j_1 > j_2 > j_3 > \cdots$ are integers such that all the negative integers except a finite number are included in the sequence. The representation of CAR is irreducible. There exists a densely defined charge operator N which acts on a basis vector by

$$\begin{aligned} N & \quad a(u_{i_1}) \cdots a(u_{i_k}) a^*(u_{j_1}) \cdots a^*(u_{j_l})|0\rangle \\ = & \quad (l-k) a(u_{i_1}) \cdots a(u_{i_k}) a^*(u_{j_1}) \cdots a^*(u_{j_l})|0\rangle \end{aligned}$$

The Fock space is a completion of the algebraic direct sum $\mathcal{F} = \widehat{\bigoplus}_{k\in\mathbb{Z}} \mathcal{F}^{(k)}$ where $\mathcal{F}^{(k)}$ is the subspace of charge k.

In the group $L\mathbb{T}_\theta$ of smooth loops in $\mathbb{T}_\theta$ any element is of the form $e^{2\pi iF}$ such that $F:\mathbb{R}\to\mathbb{R}$ is a smooth function and $F(\theta+2\pi)=F(\theta)+n_F$. $n_F\in\mathbb{Z}$ is the winding number of the loop. Then $f(\theta)=F(\theta)-n_F\theta/2\pi$ is invariant under the translations $\theta\mapsto\theta+2\pi$ and thus it can be expanded as a Fourier series $f=\sum f_k e_k$, where f_k are the Fourier coefficients for all $k\in\mathbb{Z}$. Since f is real valued these satisfy $\overline{f_k}=f_{-k}$. We can write $L\mathbb{T}_\theta=SL\mathbb{T}_\theta\times C\mathbb{T}_\theta$ such that the charge subroup $C\mathbb{T}_\theta$ consists of the group elements $e^{2\pi if_0+in_F\theta}$ and $SL\mathbb{T}_\theta$ consists of $e^{2\pi i\sum_{k\neq 0}f_k e_k}$.

The loop group $L\mathbb{T}_\theta$ is a subgroup of the restricted unitary group $U_{\text{res}}(\mathcal{H}^+ \oplus \mathcal{H}^-)$ which has a positive energy representation on a Fock space. The action of $U_{\text{res}}(\mathcal{H}^+ \oplus \mathcal{H}^-)$ can be implemented on the Fock space as a projective representation such that

$$U(g)a(u)U(g^{-1}) = a(g.u), U(g)a^*(v)U(g^{-1}) = a^*(g.v)$$

for all $g \in U_{\text{res}}(\mathcal{H}^+ \oplus \mathcal{H}^-)$ and $u, v \in \mathcal{H}$. The subroup $SL\mathbb{T}_\theta$ lies in the connected component of the identity of $U_{\text{res}}(\mathcal{H}^+ \oplus \mathcal{H}^-)$ and each charge subspace is invariant under this action. The subgroup $C\mathbb{T}_\theta$ has infinitely many disconnected components labeled by n_F. If $e^{2\pi i f_0 + i n_F \theta} \in C\mathbb{T}_\theta$, then its unitary positive energy representation is of the form

$$U(e^{2\pi i f_0 + i n_F \theta}) = e^{\pi i f_0 N} S^{n_F} e^{\pi i f_0 N},$$

where S is a shift operators which sends each charge subspace $\mathcal{F}^{(k)}$ to $\mathcal{F}^{(k+1)}$, that is, $SNS^{-1} = N - 1$.

The positive energy representation of $L\mathbb{T}_\theta$ are projective: there is a group 2-cocycle $c : L\mathbb{T}_\theta \times L\mathbb{T}_\theta \to \mathbb{T}$ such that the unitary representation satisfies

$$U(e^{iF})U(e^{iG}) = U(e^{i(F+G)})c(e^{iF}, e^{iG}).$$

We denote by $\mathcal{PF}$ the projective Fock space $\mathcal{F}/\mathbb{T}$. Then U defines a representation of $L\mathbb{T}_\theta$ on $\mathcal{PF}$.
Next we consider the Fock space theory associated to the families index problem. Fix a complex line bundle λ over M and a cover $\{\mathcal{V}_i\}$ of M which trivializes λ. Then λ is extended on the space $\mathbb{T} \times M$ so that the new transition functions satisfy $h_{ab}(\phi, x) = h_{ab}(x)$ for all $(\phi, x) \in \mathbb{T} \times M$. Similarly we can extend λ to $\mathbb{R} \times M$.

In the Fock space model, the rotations around the circle $\mathbb{T}_\phi$ in the positive direction raises the charge of the Fock bundle over $\mathbb{T} \times M$ by one. A subbundle of charge k states is of topological type $\lambda^{\otimes k}$ over M and therefore we need to introduce an operator S which creates a bundle λ from the vacuum. This process is defined only up to a phase and therefore we start by considering a projectivization of the Fock bundle over the covering space $\mathbb{R} \times M$ which we denote by $\mathbf{PF}_0$. In this case we have an operator family $S : M \to PU(\mathcal{H})$. Then we define

$$\mathbf{PF} = \mathbf{PF}_0 / \sim$$

where $\sim$ is the equivalence relation $(\phi, x, \Psi) \sim (\phi', x', \Psi')$ if and only if $\phi' = \phi + 2\pi n$ in $\mathbb{R}$, $x = x'$ in M and $\Psi' = S_x^n \Psi$ in the fibre for all $n \in \mathbb{Z}$. Then $\mathbf{PF}$ is a projective Fock bundle on $\mathbb{T} \times M$ and its cohomology class is determined by a lift of the transition functions S to unitary operators.

We pullback **PF** to the covering space $\mathbb{R} \times M$. The Dixmier-Douady class trivializes on the covering and therefore we can fix the phases to define a Hilbert bundle **F** with a structure group $U(\mathcal{F})$ on $\mathbb{R} \times M$. One uses the operation S pulled to the covering space to glue the fibres together in the transitions in the positive direction at each $2\pi\mathbb{Z}$ in $\mathbb{R}$. The vacuum of the above construction can be further twisted by a complex vector bundles of finite rank. Let ξ denote a rank n vector bundle over M, extended trivially to $\mathbb{R} \times M$. Replace next the Fock bundle **PF** by $\mathbf{P}(\mathbf{F} \otimes \xi) = \mathbf{PF}_\xi$ and **F** by $\mathbf{F} \otimes \xi = \mathbf{F}_\xi$. Define now the action of the shift operator S on the tensor product as $S \otimes 1$.

Twisted K-Theory on $\mathbb{T} \times M$

Consider a real Clifford $*$-algebra, $\mathrm{cl}(L\mathbb{T})$ generated by $\psi_n, n \in \mathbb{Z}$ subject to the relations

$$\{\psi_n, \psi_m\} = 2\delta_{n,-m}, \psi_n^* = \psi_{-n}.$$

We can fix an irreducible vacuum representation of $\mathrm{cl}(L\mathbb{T})$ such that the circle group $\mathbb{T}$ acts on the vacuum η_0 by the identity homomorphism. The operators ψ_i with $i < 0$ annihilate the vacuum and the vectors ψ_i with $i > 0$ are used to generate the basis from the vacuum subspace. We fix the sign of ψ_0 such that $\psi_0\eta_0 = \eta_0$. We denote by $\mathcal{H}_s$ this representation.

On the parameter space $\mathbb{R} \times M$ we define a trivial infinite dimensional spinor bundles $\mathbf{S} = \mathcal{H}_s \times \mathbb{R} \times M$. This is pushed down to a trivial bundle over $\mathcal{H}_s \times \mathbb{T} \times M$, to be denoted by the same symbol $\mathbf{S}$. Then we form a PU-bundle $\mathbf{P}(\mathbf{S} \otimes \mathbf{F}_\xi)$ over $\mathbb{T} \times M$. We also have the Hilbert bundle $\mathbf{S} \otimes \mathbf{F}_\xi$ over $\mathbb{R} \times M$.

We define a family of supercharge operators $Q : \mathbb{R} \times M \to \mathbf{Fred}^{(1)}(\mathbf{S} \otimes \mathbf{F}_\xi)$ coupled to a constant potential $y \in \mathbb{R}$ by

$$Q_y = \sum_{i \in \mathbb{Z}} \psi_n \otimes e_{-n} + y\psi_0 \otimes 1$$

where the operators e_n define a projective unitary representation of the loop algebra $\mathfrak{lt}$ (= Lie algebra of $L\mathbb{T}$) on $\mathbf{F}$. More precisely we can write

$$e_n = \sum_i : a^*(v_{n+i}) a(v_i) : .$$

These operators are globally defined. Initially we need to fix a phase from the twisting bundle λ to make $a^*(v_n)$ and $a(v_m)$ well-defined but since the first one is linear whereas the second one is antilinear these phases cancel each other. The usual normal ordering $::$ is applied to make the operators well defined on the Fock spaces; that is, $: a^*(v_n)a(v_m) := -a(v_m)a^*(v_n)$ if $n = m < 0$ and ordering unchanged otherwise. Q is an unbounded self adjoint operator. Its square is the operator

$$Q_y^2 = \sum_{n>0} n\psi_n\psi_{-n} + 2\sum_{n>0} e_n e_{-n} + e_0^2 + 2ye_0 + y^2 \equiv l_0^s + l_0^f + (e_0 + y)^2$$

The operators l_0^s and l_0^f are positive with zero modes corresponding to the Hilbert space sections S^n(vacuum) for any $n \in \mathbb{Z}$. This follows from $[l_0^f, S] = [l_0^s, S] = 0$. The operator e_0 counts the fermion number and thus $S^{-1}e_0 S = e_0 + 1$ and so

$$Q_y^2 S^n(\eta_0 \otimes |0\rangle) = (n + y)^2 S^n(\eta_0 \otimes |0\rangle)$$

The zero modes are localized on the submanifolds with $y \in \mathbb{Z} \subset \mathbb{R}$.

The operator S acts on the supercharge by conjugation such that $SQ_yS^{-1} = Q_{y-1}$. Therefore, if we set $y = \phi/2\pi$, then the zero modes are located on the submanifolds with $\phi \in 2\pi\mathbb{Z}$ and $SQ_{\phi/2\pi}S^{-1} = Q_{(\phi/2\pi)-1}$. This operator family can be realized as a locally defined family over $\mathbb{T} \times M$, $Q^i : \mathcal{U}_i \to \mathbf{Fred}^{(1)}$, patched together by an adjoint action of a Čech-cocycle which corresponds to the Dixmied-Douady class σ. We conclude:

Theorem *The operator family Q defines a class in the twisted K-group* $K^1(\mathbb{T} \times M, \sigma)$.

When the nontwisted groups $K^*(M)$ are known, one can use the Mayer-Vietoris sequence to study the K-theory on $\mathbb{T} \times M$ twisted by a decomposable 3-cohomology class. The base space $\mathbb{T} \times M$ is a union of $\overline{\mathbb{T}}_+ \times M$ and $\overline{\mathbb{T}}_- \times M$ where $\overline{\mathbb{T}}_\pm$ denote the closures of $\mathbb{T}_\pm$. The gerbe corresponding to the decomposable cohomology class trivializes after the circle is cut.

Therefore, we get the Mayer-Vietoris sequence

$$
\begin{array}{ccccc}
K^0(\mathbb{T} \times M, \sigma) & \xrightarrow{c_0} & K^0(\overline{\mathbb{T}}_+ \times M) \oplus K^0(\overline{\mathbb{T}}_- \times M) & \xrightarrow{a_0} & K^0(\overline{\mathbb{T}}_{+-} \\
\uparrow b_1 & & & & \downarrow b_0 \\
K^1(\overline{\mathbb{T}}_{+-} \times M) & \xleftarrow{a_1} & K^1(\overline{\mathbb{T}}_+ \times M) \oplus K^1(\overline{\mathbb{T}}_- \times M) & \xleftarrow{c_1} & K^1(\mathbb{T} \times
\end{array}
$$

Thus, there are the following group isomorphism

$$\begin{aligned} K^{*+1}(\mathbb{T} \times M, \sigma) &\simeq (K^*(\overline{\mathbb{T}}_{+-} \times M)/\mathrm{Im}(a_*)) \oplus_\zeta \mathrm{Im}(c_{*+1}) \\ &\simeq (K^*(M)^{\oplus 2}/\mathrm{Im}(a_*)) \oplus_\zeta \mathrm{Ker}(a_{*+1}) \end{aligned}$$

which is a group extension of $\mathrm{Ker}(a_{*+1})$ by $K^*(M)^{\oplus 2}/\mathrm{Im}(a_*)$ associated to some cocycle ζ in the group cohomology. In general it is impossible to fix ζ from the Mayer-Vietoris sequences and some other methods neet to be applied.

As we have seen, we need to apply coordinate transformation which corresponds to a tensor product operation by the bundle λ over M when we transform from $\overline{\mathbb{T}}_- \times M$ to $\overline{\mathbb{T}}_+ \times M$ in $\overline{\mathbb{T}}^{(1)}_{+-}$. Consider a class $(x, y) \in K^*(\overline{\mathbb{T}}_+ \times M) \oplus K^*(\overline{\mathbb{T}}_- \times M)$. The gluing maps a_* are defined by

$$a_*(x, y) = (x - y, x - y \otimes \lambda)$$

where the first component on the right side is a group element in $K^*(\overline{\mathbb{T}}^{(-1)}_{+-} \times M)$ and the second in $K^*(\overline{\mathbb{T}}^{(1)}_{+-} \times M)$. The tensor product is defined by the usual ring structure in the ordinary K-theory.

Homotopy equivalence of K-theory gives

$\mathrm{K}^*(\overline{\mathbb{T}}_{+-} \times M) \simeq K^*(M)^{\oplus 2}, K^*(\overline{\mathbb{T}}_{\pm} \times M) \simeq K^*(M)$.We obtain

$$K^*(M)^{\oplus 2}/\mathrm{Im}(a_*) = K^*(M)/K^*(M) \otimes (1 - \lambda)$$

and in the case when λ is nontrivial and nontorsion

$$\mathrm{Im}(c_{*+1}) = \mathrm{Ker}(a_*) = 0$$

Theorem *When* λ *is a nontrivial nontorsion complex line bundle the abelian groups* $K^*(\mathbb{T} \times M, \sigma)$ *are isomorphic to* $K^{*-1}(M)/(K^{*-1}(M) \otimes (1-\lambda))$. *In the general case when* λ *is nontrivial,* $K^*(\mathbb{T} \times M, \sigma)$ *is an extension of the group*

$$\{x \in K^*(M) | x = x \otimes \lambda\} \text{ by } K^{*-1}(M)/(K^{*-1}(M) \otimes (1-\lambda)).$$

For example, when $M = S^2$ is the unit sphere and λ is the complex line bundle equal to k:th tensor power of the generator, one obtains the known result $K^1(\mathbb{T} \times M, \sigma) = \mathbb{Z} \oplus \mathbb{Z}_k$ and $K^0(\mathbb{T} \times M, \sigma) = 0$.
For $M = \mathbb{T}^2$ the corresponding groups are $\mathbb{Z} \oplus \mathbb{Z}_k$ and $\mathbb{Z}^2$.
Torsion in λ makes things more complicated. For example, if $p\lambda = 0$ then $x = x \otimes \lambda$ when x is a trivial vector bundle of rank p.

K-Theory Class from Superconnection Analysis

We study a superconnection associated to the family Q over the covering space $\mathbb{R} \times M$. For this we introduce a scaling parameter t, however, the cohomology class determined by the superconnection is independent of t. Therefore we take the limit $t \to \infty$ where the superconnection gets a simple form. The connection $\nabla = \nabla_M \otimes 1 + 1 \otimes \nabla_\xi$ consists of of a connection ∇_ξ of the bundle ξ over M and a connection ∇_M of the twisting line bundle λ over M. The action of the connection ∇_M on the fermion number n sector is the n'th tensor power of the connection in the line bundle λ; in particular, on the vacuum sector the only nontrivial piece is ∇_ξ. Let us define

$$D_t = \sqrt{t}\chi Q + \nabla$$

We write locally $\nabla = d + \omega$ where ω is the matrix valued connection form acting on the sections of the Fock bundle and $\hat{F} = \nabla^2$ is the curvature two form, composed of $\beta_M = \nabla_M^2 = e_0\beta_M$ and $F_\xi = \nabla_\xi^2$. The formal symbol χ with $\chi^2 = 1$ is introduced since the Clifford algebra of the loop group on the circle is odd (the circle is odd dimensional).
The symbol χ is defined to commute with Q and anticommute with odd differential forms. Note that the Bismut superconnection for families of Dirac operators contains a term proportional to the curvature with a factor $1/\sqrt{t}$. The motivation for that term is that in the limit $t \to 0$ one obtains from the character formula below the local Atiyah-Singer index formula. However, here we shall study the limit $t \to \infty$ and we drop this term. We have

$$d = d_y + d_M, \hat{F} = d\omega + \omega^2 = e_0\beta_M \otimes 1 + 1 \otimes F_\xi$$

(we denote $y = \phi/2\pi$). The square of the superconnection is

$$D_t^2 = tQ_y^2 + \sqrt{t}\chi(-dQ_y + [Q_y, \omega]) + \hat{F}.$$

The following holds in this case

$$-dQ_y + [Q_y, \omega] = -\psi_0 dy.$$

Then

$$D_{t,y}^2 = tQ_y^2 - \sqrt{t}\chi\psi_0 dy + \hat{F}.$$

The translation by 1 in $\mathbb{R}$ has the effect of operation S in the fibres of the Fock bundle. Then using $e_0 S^{-1} = S^{-1}(e_0 - 1)$ and $Q^2_{y+1} = S^{-1} Q^2_y S$ one gets

$$D^2_{t,y+1} = S^{-1} D^2_{t,y} S + \beta_M.$$

Now if we define the superconnection character form by

$$\Theta_y = \mathrm{sTr}(e^{-D^2_{t,y}}),$$

where the supertrace sTr picks up the terms linear in χ. Then

$$\Theta_{y+1} = \mathrm{sTr}(e^{-D^2_{t,y} - \beta_M}) = \Theta_y \wedge e^{-\beta_M}.$$

Next thing is to push these classes to the cohomology of $\mathbb{T}_\phi \times M$ by the standard map $f : \mathbb{R} \times M \to \mathbb{T} \times M$ (sends the coordinate to the angle variable). There are now two ways to go: we can define new forms

$$\tilde{\Theta}_y = e^{y\beta_M} \wedge \Theta_y,$$

so that $\tilde{\Theta}_{y+1} = \tilde{\Theta}_y$ and then study the usual twisted cohomology:

$$(d - H)f_*\tilde{\Theta}_y = 0$$

for $H = dy \wedge \beta_M$. In the twisted cohomology $f_*\tilde{\Theta}$ is a cocycle. Alternatively, we can study the forms Θ on $\mathbf{R} \times M$ pushed to the cohomology $H^*(\mathbb{T}_\phi \times M)/\langle \beta_M \rangle$. The forms Θ are indeed periodic modulo $\beta_M \wedge \Theta$.

Put $D^2_{t,x} = t(Q^2 + K_t)$. We use the Volterra series

$$\Theta_y = \mathrm{sTr}\Big(e^{-tQ^2} + \sum_{n\geq 1}(-t)^n \int_{\triangle_n} e^{-ts_1 Q^2} K_t e^{-ts_2 Q^2} \cdots e^{-ts_n Q^2} K_t e^{-ts_{n+1} Q^2} ds_1 ds_2 \cdots ds_{n+1}\Big).$$

The operator families $K_t = (1/t)\hat{F} - (\chi/\sqrt{t})\psi_0 dy$ commute with Q^2 and we can simplify

$$e^{-ts_1 Q^2} K_t e^{-ts_2 Q^2} \cdots e^{-ts_n Q^2} K_t e^{-ts_{n+1} Q^2} = K_t^n e^{-tQ^2}$$

The forms χdy and $\hat{F}$ commute. Thus,

$$K_t^n = \sum_{k=0}^{n} \binom{n}{k} \Big(\frac{-\chi\psi_0 dy}{\sqrt{t}}\Big)^{n-k} \wedge \Big(\frac{\hat{F}}{t}\Big)^k = -n \frac{\chi\psi_0 dy}{\sqrt{t}} \wedge \Big(\frac{\hat{F}}{t}\Big)^{n-1} + \Big(\frac{\hat{F}}{t}\Big)^n.$$

As the volume of an n-simplex is $1/n!$ we get

$$
\begin{aligned}
\Theta_y &= \mathrm{sTr}\Big(e^{-tQ^2} + \sqrt{t}\sum_{n\geq 1}\frac{\chi\psi_0 dy \wedge (-\hat{F})^{n-1}}{(n-1)!}e^{-tQ^2} \\
&+ \sum_{n\geq 1}\frac{(-\hat{F})^n}{n!}e^{-tQ^2}\Big) \\
&= \mathrm{Tr}\Big(\sqrt{t}\sum_{n\geq 1}\frac{\psi_0 dy \wedge (-\hat{F})^{n-1}}{(n-1)!}e^{-tQ^2}\Big).
\end{aligned}
$$

Next recall that

$$Q^2 = l_0^s + l_0^f + (e_0 + y)^2.$$

We use the asymptotic expansion for the positive operator $e^{-t(l_0^s + l_0^f)}$ as $t \to \infty$. In this limit, the operator e^{-tQ^2} converges to zero outside the subspace with vacuum state in the fermionic sector. The following formulas hold for the Dirac measure

$$\delta(\phi - a) = \lim_{t\to\infty} \sqrt{\frac{t}{\pi}} e^{-t(\phi-a)^2}, \ \lim_{t\to\infty} \frac{1}{t^p}\sqrt{\frac{t}{\pi}} e^{-t(\phi-a)^2} = 0 \text{ with } p \in \mathbb{N}.$$

Therefore

$$\lim_{t\to\infty} \Theta_y = \sqrt{\pi}\mathrm{Tr}\Big(\psi_0 P \delta(e_0 + y) e^{-\hat{F}}\Big) = \sqrt{\pi}\delta(e_0 + y)\mathrm{tr}_\xi(e^{-\hat{F}})$$

where P denotes the projection onto the fermionic vacuum subspace and $\delta(e_0 + y)$ denotes the Dirac delta distribution. The form Θ_y then localizes at the points in $\mathbb{Z} \subset \mathbb{R}$.

To get integral cohomology classes we set a normalization function

$$\varphi : \Lambda_{\mathbb{C}}(M) \to \Lambda_{\mathbb{C}}(M), \varphi(\Omega) = (2\pi i)^{-\frac{\deg(\Omega)}{2}}\Omega.$$

Now we push the form $\varphi \lim_{t\to\infty} \Theta_y$ to $\mathbb{T} \times M$ and study its equivalence class modulo $\beta_M/2\pi i \wedge \varphi \lim_{t\to\infty} \Theta$.
The analysis above proves that the twisted K^1-theory class associated to the family Q and the vacuum vector bundle ξ are distinguished by the Chern character of $-F_\xi$ evaluated in the quotient.

Consider the case $\dim(M) = 2$, then the cohomology class associated to the superconnection gives $\sqrt{\pi}$ times $\delta(e_0 + y)$ times

$$\mathrm{rk}(\xi) - \frac{\mathrm{tr}_\xi(F_\xi)}{2\pi i} - y\,\mathrm{rk}(\xi)\frac{\beta_M}{2\pi i}$$

where $\mathrm{tr}_\xi(F_\xi)$ in the case dim $M = 2$ is an integer n times the curvature F_b of the basic line bundle over M. Now if we push this form to $\mathbb{T} \times M$ and study its equivalence class modulo $\beta_M/2\pi i \wedge \varphi \lim_{t\to\infty} \Theta$, then, in this cohomology the superconnection gives the component in 3-cohomology

$$-\frac{n\sqrt{\pi}dy \wedge F_\xi}{2\pi i} \mathrm{mod} \frac{\sqrt{\pi}dy \wedge \beta_M}{2\pi i}.$$

Therefore this method can be used to separate different twisted K-theory classes.

In the case $M = S^2$ or $M = \mathbb{T}^2$ and $F_\lambda = kF_b$ and $F_\xi = nF_b$, the operator family defines a twisted K^1-group element $n \oplus \mathrm{rk}(\xi)$ in $K^1(\mathbb{T} \times S^2, dy \wedge kF_b) = \mathbb{Z}_k \oplus \mathbb{Z}$, as can be computed from the Mayer-Vietoris sequence.

P. Donovan and M. Karoubi: Graded Brauer groups and K-theory with local coefficients. Inst. Hautes tudes Sci. Publ. Math. No. 38 (1970), 5 - 25.

J. Dixmier and A. Douady: Champs continus d'espaces hilbertiens et de $C*$ -algbres. Bull. Soc. Math. France **91,** (1963), 227 - 284.

N. Kuiper: The homotopy type of the unitary group of Hilbert space. Topology **3,** (1965), 19 - 30.

G. Segal and M.F. Atiyah: Twisted K-theory. Ukr. Mat. Visn. **1** (2004), no. 3, 287–330; translation in Ukr. Math. Bull. **1** (2004), no. 3, 291 - 334.

J.-M. Bismut: Localization formulas, superconnections, and the index theorem for families. Comm. Math. Phys. **103** (1986), no. 1, 127 - 166

- A. L. Carey, J. Mickelsson, and Bai-Ling Wang: Differential twisted K-theory and applications. J. Geom. Phys. **59** (2009), no. 5, 632 - 653.
- J. Rosenberg: Continuous-trace algebras from the bundle theoretic point of view. J. Austral. Math. Soc. Ser. **A 47** (1989), no. 3, 368 - 381.
- D. Freed, M. Hopkins, and C. Teleman: Twisted equivariant K-theory with complex coefficients. J. Topol. **1** (2008), no. 1, 16 - 44.
- A. Carey, J. Mickelsson, and M. Murray: Index theory, gerbes, and Hamiltonian quantization. Comm. Math. Phys. **183** (1997), no. 3, 707 - 722,
- John Lott: Higher-degree analogs of the determinant line bundle. Comm. Math. Phys. **230** (2002), no. 1, 41 - 69.

- P. Bouwknegt, A. Carey, V. Mathai, M. Murray, and D. Stevenson: Twisted K-theory and K-theory of bundle gerbes. Comm. Math. Phys. **228,** (2002), no. 1, 17 - 45.
- J. Mickelsson: Gerbes, (twisted) K-theory, and the supersymmetric WZW model. Infinite dimensional groups and manifolds, 93 - 107, IRMA Lect. Math. Theor. Phys., **5,** de Gruyter, Berlin, 2004.
- V. Mathai, R.B. Melrose, and I.M. Singer: The index of projective families of elliptic operators: the decomposable case. Asterisque, **328,** 255 - 296, 2009. arXiv:0809.0028
- J. Mickelsson and J.-P. Pellonpää: Families Index Theorem in Supersymmetric WZW Model and Twisted K-Theory: The SU(2) Case. Comm. Math. Phys. **271,** (2007), 775 - 789.
- A. Pressley and G. Segal: *Loop Groups.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.