LECTURE 1: THE CHIRAL ANOMALY: EUCLIDEAN FIELD THEORY

1.1. Historical remarks

The Dirac equation for massless fermions has the chiral symmetry $\psi(x) \mapsto e^{i\alpha\Gamma}\psi(x)$ where Γ is the chirality operator in the Clifford algebra (in even dimensions $2n$) anticommuting with the generators γ_i $(i = 1, 2, \ldots, 2n)$. According to Noether's theorem, there is an associated conserved current corresponding to the symmetry; in this case the current conservation law reads

$$\partial^\mu \overline{\psi}\Gamma\gamma_\mu\psi = 0.$$

If the fermion field has a mass there is an additional term on the right-hand-side. In particular, in an important QCD application (where the discussion on the chiral anomaly started from) the above equation is written as

$$\partial^\mu \overline{q}\Gamma\gamma_\mu\tau_3 q = 2m(\overline{q}\Gamma\tau_3 q) \equiv F_\pi m_\pi^2 \pi^0$$

where π^0 is the neutral component of the pion field and F_π is a coupling constant. This is called the partially conserved axial current conservation law (PCAC) in QCD.

However, it was realized by perturbative calculations in quantum field theory that the above equation is not quite true; there is *an anomalous term* on the right-hand-side of the equation. The correct equation is

$$\partial^\mu \overline{q}\Gamma\gamma_\mu\tau_3 q = 2m(\overline{q}\Gamma\tau_3 q) \equiv F_\pi m_\pi^2 \pi^0 - \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

where F is the field strength of the electromagnetic field. This is often called the ABJ-anomaly, after Stephen Adler, Phys. Rev. 177, p.2426, 1969; John Bell and Roman Jackiw, Nuovo Cimento A 60, p.47, 1969 or the triangle anomaly since it comes from the triangle Feynman graph (draw a picture!) with an outgoing pion field and two outgoing lines for the electromagnetic field.

Later it was also realized in perturbative QFT calculations that there are problems with *nonabelian gauge symmetry,* (Roman Jackiw and Kenneth Johnson, Phys. Rev. 182, p.1459, 1969; W.A. Bardeen, Phys. Rev. 184, p.1848, 1969). In the nonabelian case the currents are labelled by an extra index $a = 1, 2, \ldots N$ where N is the dimension of the nonabelian gauge group. Since in the nonabelian case the source for the anomalous current turns out to be proportional to

$$\operatorname{tr} T_a F \wedge F$$

it depends on an extra factor on the right-hand-side which is

$$\operatorname{tr} T_a(T_b T_c + T_c T_a) \tag{1.1.1}$$

where the T_a's are the generators of the gauge group G acting through a given unitary representation on the components of the fermion fields. Physicist want that the theory is anomaly free, so the vanishing of the above expression becomes crucial. For example, if $G = SU(2)$ or $G = SO(n)$ this is always true. However, for the important case of QCD the gauge group is $SU(3)$ and the anomaly cancellation depends on the representation; the condition (1.1.1) turns out to be satisfied in the case of the standard model of strong and electro-weak interactions.

The appearance of the form $F \wedge F$ (the second term in the expansion of the Chern character) in the anomaly equations points out that the anomaly has something to do with index theory of Dirac operators. In the case of the abelian chiral anomaly, this link to index theory was clarified in the 70's (N.K. Nielsen, H. Römer, B. Schroer, Nucl. Phys. B 127, p.493, 1978; Phys. Lett. 70B, p.445, 1977). In the case of the nonabelian gauge anomaly the relation to the families index theory was clearly formulated in a short paper by M.F. Atiyah and I.M. Singer in Proc. Natl. Acad. Sci (USA) 81, p. 2597, 1984.

1.2. The nonabelian anomaly in the euclidean formulation of QFT

Consider a Dirac field ψ on a compact even dimensional Riemannian spin manifold M coupled to a vector potential A (a connection in a complex vector bundle E over M). We want to make sense of the *effective action* $S_{eff}(A)$ of the Dirac operator D_A in the background gauge field A. What is the effective action? Consider a Lagrangian $L(\psi, A)$ depending in some way of a field ψ (to be quantized) and an external nonquantized field A. The effective action is then formally defined by

$$e^{-S_{eff}(A)} = \int_\psi e^{-L(\psi,A)}.$$

In the case when L is quadratic in ψ one can make sense of the integral. First, suppose that $L = (\psi, T\psi)$ where T is a positive linear operator and ψ is a vector in a n-dimensional vector space. Then the Gaussian integral is well-defined and equal to

$$I(T) = \prod \sqrt{\frac{\pi}{\lambda_i}} = (\pi)^{n/2} \cdot (\det T)^{-1/2}. \tag{1.2.1}$$

In quantum theory the Dirac field ψ is supposed to describe fermions. This means that the components of the spinor ψ should *anticommute* among themselves instead of commuting like the components of ordinary vectors. Thus the correct finite-dimensional analog of (1.2.1) is something like

$$\int_{\psi^*, \psi \in \mathbb{C}^n} e^{-\psi^* T \psi} \tag{1.2.2}$$

where ψ_i^*, ψ_i are elements of a Grassmann algebra. The top element (element of highest degree) in the Grassmann algebra is $\psi_1^* \dots \psi_n^* \psi_1 \dots \psi_n$. We shall *define* the integral (1.2.2) as the coefficient of the top term in the expansion of the exponential; this is

$$\begin{aligned} \frac{(-1)^n}{n!} \sum (\psi_{i_1}^* T_{i_1 j_1} \psi_{j_1}) \dots (\psi_{i_n}^* T_{i_n j_n} \psi_{j_n}) \\ &= -\sum \psi_1^* \dots \psi_n^* T_{1 j_1} \dots T_{n j_n} \psi_{j_1} \dots \psi_{j_n} \\ &= -\det(T)\, \psi_1^* \dots \psi_n^* \psi_1 \dots \psi_n. \end{aligned}$$

Motivated by the finite-dimensional example we define the effective action $S_{eff}(A)$ to be the determinant of the Dirac operator. However, we still have to define what we mean by $\det(D_A)$. The determinant of a linear operator T in a Hilbert space is well-defined if $T = 1 + S$ where S is a trace-class operator, i.e., the sum of the eigenvalues λ_i of S form an absolutely convergent series. In that case the determinant of T is $\prod(1 + \lambda_i)$ and this converges. In the case of a Dirac operator the eigenvalues do not even form a bounded sequence.

We must introduce a *regularized determinant*. There is a variety of ways to regularize the infinite product of the eigenvalues. We would like to have some sort of continuity of the determinant: The determinant should be expressible as a function $f(\lambda)$ of the set of eigenvalues $\lambda = \{\lambda_k\}$ of D_A such that f is continuous in each argument λ_i (of course f must be such that its value does not depend on the order of the arguments). In addition, we require that f is zero if and only if one of the eigenvalues λ_i vanishes. One simple choice for f is the "cutoff regularization" of the determinant,

$$\det{}^{(M)}(D_A) = \prod_{|\lambda_i|<M} \lambda_i,$$

where $M > 0$ is a "large" cutoff parameter. The obvious disadvantage of this determinant is that it gives no information about the large part of the spectrum. One can introduce the cutoff in a smoother way by choosing a function h on the reals such that $h(x) = x$ for $|x| < M$ and $h(x) \to 1$ very fast as $|x| \to \infty$. A regularized determinant can then be defined as

$$\det{}^h(D_A) = \prod h(\lambda_i).$$

For an elliptic differential operator on a compact manifold such that the real parts of the eigenvalues λ_i are bounded from below (e.g., the Laplace operator) there is a slightly more sophisticated (and more symmetric) way to define the determinant, by the so-called ζ function regularization. Consider the function

$$\zeta(s, \lambda) = \sum_{Re\lambda_i>0} \lambda^{-s}$$

One can show that this function is holomorphic in the half-plane $\mathrm{Re}\, s > s_0$ for large enough s_0. Furthermore, it extends holomorphically to a regular function at the point $s = 0$. Let $\lambda_1, \ldots, \lambda_p$ be the set of eigenvalues with $\mathrm{Re}\, \lambda_i \leq 0$. The ζ-regularized determinant is then defined as

$$\lambda_1\lambda_2 \ldots \lambda_p \exp\left[-\frac{d}{ds}\zeta(s, \lambda)\right]|_{s=0} \tag{1.2.3}$$

It is a simple exercise to show that in the finite-dimensional case (1.2.2) gives the usual determinant.

1.3. The determinant from weighted traces

Formally,

$$\log\det(D_A) = \mathrm{tr}\, \log(D_A).$$

Of course, the trace is ill-defined. We can slightly improve the situation by considering the relative determinant

$$\log\det(D_B^{-1}D_A) = \mathrm{tr}\ \log(D_B^{-1}D_A)$$

for some fixed background connection. In the case of a trivial vector bundle E a simple choice is $B=0$ as a Lie algebra valued 1-form on the base M. Then

$$D_B^{-1}D_A = 1 + D_0^{-1}A$$

and we can expand the right-hand-side as a power series in the pseudo-differential operator $D_0^{-1}A$. In the case D_0 is not invertible we can slightly perturbe it to make it invertible. Now the the powers $(D_0^{-1}D_A)^k$ are trace-class when $k > \dim M$.

Remark The powers $(D_0^{-1}D_A)^k$ in the logarithmic expansion are the *one loop Feynman graphs* with k external legs and the internal fermion propagators correspond to the Green's function D_0^{-1}. In four space-time dimensions the graphs with more than 4 external legs are all converging Feynman integrals but the lower graphs need an (infinite) renormalization.

In order to deal with the exponents k smaller or equal to the dimension of M we introduce the *weighted trace* (S. Paycha, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 4 (2001), no. 2, 221- 266). For that purpose fix a positive order pseudodifferential operator Q with a spectral cut along a ray starting from the origin, so that we can define the powers Q^{-z} for complex numbers z. For any given pseudodifferential operator T the trace

$$\mathrm{tr}\, Q^{-z}T$$

is well-defined when the real part of z is large enough and can be continued to a holomorphic function in an neighborhood of the point $z=0$, with a single pole at $z=0$. [For the continuation to a neighborhood of $z=0$ one can replace the trace above by the Kontsevich-Vishik canonical trace (see also PhD thesis of L. Friedlander)]. Then the finite part

$$\mathrm{f.p.}_{z=0}\mathrm{tr}\, Q^{-z}T$$

is called the weighted trace of T to be denoted $\mathrm{TR}^Q T$. The singular part at $z=0$ is equal to

$$\frac{1}{qz}\mathrm{res}(T)$$

and can be take as the definition of the operator residue (Guillemin-Wodzicki residue) $\mathrm{res}(T)$. A local expression for the residue is

$$\mathrm{res}(T) = \frac{1}{(2\pi)^n}\int_M dx \int_{|p|=1} \mathrm{tr}\, t_{-n}(x,p)dp \tag{1.3.1}$$

where t_k is the term in the asymptotic expansion of the symbol of T which is homogeneous in the momenta p of degree k. The most fundamental property of the residue is that it is trace in the sense that

$$\mathrm{res}[S,T] = 0$$

for any pair of pseudodifferential operators. Another important property is

$$\mathrm{TR}^Q[S,T] = \frac{1}{q}\mathrm{res}S[\log(Q),T]. \tag{1.3.2}$$

We can now apply this to the case of the Dirac operator to define

$$S_{eff} = \mathrm{TR}^Q \log(D_0^{-1}D_A) \tag{1.3.3}$$

and we can take for example $Q = |D_0|$. The result does depend on the choice of Q. However, the good news is that

$$\mathrm{TR}^Q(T) - \mathrm{TR}^P(T) = -\mathrm{res}\, T(\frac{\log Q}{q} - \frac{\log P}{p}) \tag{1.3.4}$$

where q is the order of Q and p is the order of P. The operator residue is a local expression in the sense that it depends only on the integral of the term in the asymptotic expansion which is homogeneous in the momenta of order $-n$, $n = \dim M$. For this reason, changing the choice of the 'renormalization', the choice of Q, changes the effective action by a finite differential polynomial in the external fields.

One would also like to make sense of the effective action in the case of *chiral fermions* , that is, one considers the Weyl fermions, the left-handed components of the Dirac fermions. However, there is a problem since the Dirac-Weyl operator in the massless case is a map D_A^+ from the left-handed components to the right-handed components (eigenspaces of Γ with eigenvalues ± 1). Thus the determinant is ill-defined. To fix this one could fix a background connection and consider the determinant of the operator $(D_B^+)^* D_A^+$ which is a linear operator from left-handed to left-handed components; this is the method of Atiyah-Singer (1984). Alternatively, we can use the log-det formula and insert a projection P_- onto the left-handed components, i.e., define

$$S_{eff}(A) = \mathrm{tr}\, P_- \log(D_B^{-1}D_A). \tag{1.3.5}$$

It turns out that for the discussion of the gauge anomaly, the slightly more symmetric form

$$S_{eff}(A) = \mathrm{str}\log(D_B^{-1}D_A),$$

where $\mathrm{str}(T) = \mathrm{tr}(\Gamma T)$, is more convenient and leads to an equivalent result.

Next we consider the gauge variation

$$\mathcal{L}_X S_{eff}(A) = \omega(A;X) \tag{1.3.6}$$

with $\mathcal{L}_X A = [A,X] + dX$ for an infinitesimal gauge transformation X. For simplicity, we shall restrict to the case of a trivial bundle E so the forms A are globally defined on M and we can take $B=0$ and $X : M \to \mathfrak{g}$ a smooth map, where $\mathfrak{g}$ is the Lie algebra of G. Denote $\not{A} = A^i\gamma_i, \not{d}X = \gamma^i\partial_i X$. Then, after some algebraic rearrangements of the terms in (1.3.5) one can show that ω is a weighted trace of commutators (with $Q = |D_0|$)

$$\begin{aligned}\omega(X;A) = \mathrm{TR}^Q[\log(1 + \frac{1}{D_0}\not{A}), iXP_-] \\ + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(k+1)} \sum_{j=0}^{k-1} \mathrm{TR}^Q \left[(\frac{1}{D_0}\not{A})^j(\frac{1}{D_0}\not{d}X), (\frac{1}{D_0}\not{A})^{k-j}P_- \right],\end{aligned}$$

and can therefore be computed by the residue formula (1.3.1). The final result is, up to a gauge variation of a finite differential polynomial [E. Langmann and J. Mickelsson, Lett. Math. Phys. 36, no. 1, 45- 54, 1996],

$$\omega(A;X) = \frac{(-i)^n}{(n-1)!(2\pi)^n}\int_0^1 dt \frac{t-1}{t}\int dx \,\text{Symm tr}(dX A_t F_t^{(n-2)/2}) \tag{1.3.7}$$

where $A_t = tA$ and $F_t = tdA + t^2A^2$.

This result can be derived using a families index theory argument [Atiyah-Singer, 1984]. First, the gauge algebra cocycle ω can be exponentiated to a gauge group cocycle $\Omega(A;g)$ with values in $S^1 \subset \mathbb{C}$,

$$\Omega(A;g_1g_2) = \Omega(A;g_1)\Omega(A^{g_1};g_2) \tag{1.3.8}$$

with $g : M \to G$ a smooth gauge transformation. The cocycle Ω defines a complex line bundle L over $\mathcal{A}/\mathcal{G}_0$, the moduli space of gauge connections with $\mathcal{G}_0$ the group of *based* gauge transformations. The total space of L is defined as the quotient of $\mathcal{A} \times \mathbb{C}$ modulo the right action by $\mathcal{G}$, $(A,\lambda)\cdot g = (A^g, \lambda\Omega(A;g))$. The curvature of L can be computed from the index formula as

$$c_2 = \int_M \hat{A}(R)ch(F)|_{[2n,2]} \tag{1.3.9}$$

where R is the Riemannian curvature of M (here we consider the case of a fixed metric on M) and F is the $\mathfrak{g}$-valued curvature tensor of the bundle E, viewed as a vector bundle over $M \times \mathcal{A}/\mathcal{G}$.

The pull-back π^*c, with respect to the projection $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{G}_0$, of the curvature is an exact from on the affine space $\mathcal{A}$. Thus $\pi^*c = d\omega$ for some 1-form on $\mathcal{A}$. Since it is a pull-back with respect to the canonical projection, it has to be closed along gauge directions on $\mathcal{A}$. Thus it defines a 1-cocycle of $Map(M,\mathfrak{g})$ with coefficients in the space of functions of the variable A. This 1-cocycle is the cocycle ω derived by analytic methods above.

The same formula (1.3.9) can be used also when M is odd dimensional; in that case one gets characteristics classes on $\mathcal{A}/\mathcal{G}_0$ in odd dimensions. In particular, the 3-cohomology class

$$c_3 = \int_M \hat{A}(R)ch(E)|_{[dimM,3]} \tag{1.3.10}$$

is important in Hamiltonian quantization. There it defines the Dixmier-Douady class for a projective bundle of Fock spaces over the gauge configurations.

LECTURE 2: THE HAMILTONIAN ANOMALY AND GERBES

2.1. Canonical quantization, Bogoliubov automorphisms, and central extensions

Let H be a complex Hilbert space. We consider an algebra with a unit 1, the algebra of *canonical anticommutation relations* (CAR), generated by elements $a(u), a^*(v)$ with $u, v \in H$, with the only relations

$$\begin{aligned} a^*(u)a(v) + a(v)a^*(u) &= (v, u) \\ a(u)a(v) + a(v)a(u) &= 0 \\ a^*(u)a^*(v) + a^*(v)a^*(u) &= 0 \end{aligned} \tag{2.1.1}$$

and the relations arising from the requirement that $u \mapsto a^*(u)$ is linear and $u \mapsto a(u)$ is antilinear. If H is finite-dimensional, $\dim H = n$, the CAR algebra has dimension 2^{2n}, otherwise it is infinite-dimensional.

Suppose $H = H_+ \oplus H_-$, where $H_\pm$ are closed subspaces. Let $\pi_\pm$ be the corresponding orthogonal projections. Assume that the elements of the CAR algebra are represented by linear operators in a Hilbert space $\mathcal{F}$ such that $a^*(u)$ is the adjoint of $a(u)$ for any $u \in H$. We say that this is a Fock representation with a Dirac vacuum $|0>$ if it is irreducible and there is a (normalized) vector $|0> \in \mathcal{F}$ such that

$$\begin{aligned} a(u)|0> &= 0 \text{ for all } u \in H_+ \\ a^*(v)|0> &= 0 \text{ for all } v \in H_-. \end{aligned} \tag{2.1.2}$$

The vector $|0>$ is sometimes called the 'Dirac sea'. One can prove that for each polarization $H = H_+ \oplus H_-$ there is a (up to equivalence) unique Fock representation. One has also the following theorem:

Theorem 2.1.3 (Shale-Stinespring). *Two different polarizations $H = H_+ \oplus H_- = W_+ \oplus W_-$ define equivalent Fock representations of the CAR algebra if and only if the projections $W_+ \to H_-$ and $W_- \to H_+$ are Hilbert-Schmidt.*

We skip the proofs. Concerning the basic properties of representations of CAR algebra and references to original articles we refer to the review article by H. Araki in Contemporary Mathematics, American Mathematical Society vol. 62, 1987). Let $g : H \to H$ be a unitary operator. A Bogoliubov automorphism of the CAR algebra is fixed by the conditions $a^*(v) \mapsto a^*(gv)$ and $a(v) \mapsto a(gv)$. A unitary map $T(g) : \mathcal{F} \to \mathcal{F}$ is a *implementor* of the Bogoliubov automorphism g if

$$T(g)a^*(v)T(g)^* = a^*(gv) \text{ and } T(g)a(v)T(g)^* = a(gv) \tag{2.1.4}$$

for all $v \in H$. Let us denote $\epsilon = \pi_+ - \pi_-$.

Theorem 2.1.5. *A Bogoliubov automorphism g is implementable if and only if $[\epsilon, g]$ is Hilbert-Schmidt.*

In a similar way, one defines Bogoliubov endomorphims and their implementors: now we are looking for antihermitean operators $dT(X) : \mathcal{F} \to \mathcal{F}$ such that

$$[dT(X), a^*(v)] = a^*(Xv) \text{ and } [dT(X), a(v)] = a(Xv) \tag{2.1.6}$$

where $X : H \to H$ is antihermitean. The condition for the existence of $dT(X)$ is that $[\epsilon, X]$ is Hilbert-Schmidt.

There is an explicite construction for operators $\hat{X} = dT(X)$ satisfying (2.1.6). Let $\{e_n\}$ be a basis of H such that $e_n \in H_+$ for $n \geq 0$ and $e_n \in H_-$ for $n < 0$. Denote $a_n = a(e_n)$ and $a^*_n = a^*(e_n)$. Define a normal ordering by

$$: a^*_n a_m := \begin{cases} -a_m a^*_n \text{ for } n = m < 0 \\ a^*_n a_m \text{ otherwise} \end{cases}$$

and set

$$\hat{X} = \sum X_{nm} : a^*_n a_m : \tag{2.1.7}$$

where $X_{nm} = (e_n, X e_m)$. The condition (2.1.6) is checked by a direct computation from the defining relations (2.1.1). Note that for finite matrices X

$$\hat{X} = \sum X_{nm} a^*_n a_m - \sum_{k<0} X_{kk}. \tag{2.1.8}$$

From this we get, for finite matrices,

$$[\hat{X}, \hat{Y}] = \sum X_{nm} Y_{kl} [a^*_n a_m, a^*_k a_l] = \sum [X, Y]_{ij} a^*_i a_j = \widehat{[X, Y]} + \sum_{k<0} [X, Y]_{kk}.$$

The constant term can be written as

$$c(X, Y) = \operatorname{tr} \pi_+ [X, Y] = \frac{1}{4} \operatorname{tr} \epsilon [\epsilon, X][\epsilon, Y]. \tag{2.1.9}$$

The last form is continuous in the space of matrices $X, Y \in \mathfrak{gl}_{res}$. Here the restricted linear Lie algebra $\mathfrak{gl}_{res}$ consists of bounded operators X such that $[\epsilon, X]$ is Hilbert-Schmidt. The topology is defined by the HS norm on the off-diagonal blocks and by the operator norm on the diagonal blocks. Using the continuity of c one can show that

$$[\hat{X}, \hat{Y}] = \widehat{[X, Y]} + c(X, Y) \tag{2.1.10}$$

for all $X, Y \in \mathfrak{gl}_{res}$, (L.E. Lundberg, Comm. Math. Phys. 50 (1976), no. 2, 103 - 112).

Unfortunately, the HS condition is seldom satisfied for operators of physical interest. The important cases where the condition is satisfied are local currents in $1 + 1$ -dimensional field theory models and external field scattering operators in all dimensions.

Example Let X be a multiplication operator $\psi \mapsto X(x)\psi(x)$ acting on vector valued functions ψ on the circle and let H be the Hilbert space of square-integrable functions ψ. Let $\epsilon = p/|p|$, the sign of the momentum operator on the circle. If X is smooth the symbol of the commutator $[\epsilon, X]$ is localized at $p=0$ in the momentum space and therefore defines a HS operator. In this case $c(X,Y)$ can be explicitly evaluated (Exercise!) and the result is

$$c(X,Y) = \frac{1}{2\pi i}\int \mathrm{tr}\, X'(x)Y(x)dx.$$

There is a nice way to derive the local expression for the 2-cocycle using operator residues and weighted traces. Using $\epsilon^2 = 1$ and the conditional trace $\mathrm{tr}_C(X) = \frac{1}{2}\mathrm{tr}(X + \epsilon X \epsilon)$ we first get

$$\frac{1}{4}\mathrm{tr}\,\epsilon[\epsilon, X][\epsilon, Y] = \frac{1}{2}\mathrm{tr}_C X[\epsilon, Y] = \frac{1}{2}\mathrm{tr}_C([X\epsilon, Y] - \epsilon[X, Y]).$$

The two terms on the right are not (even conditionally) trace-class but the divergences cancel and therefore we may write

$$c(X,Y) = \frac{1}{2}\mathrm{TR}^Q\,[X\epsilon, Y] - \frac{1}{2}\mathrm{TR}^Q\,\epsilon[X,Y].$$

The second term is a coboundary in Lie algebra cohomology and thus

$$c(X,Y) \sim \frac{1}{2}\mathrm{TR}^Q\,[X\epsilon, Y].$$

For a trace of a commutator of PSDO's we have an local explicit expression (we can use as a weight Q the length of the momentum $Q = |p|$ in one dimension)

$$\mathrm{TR}^{|p|}[a,b] = -\frac{1}{2}\mathrm{res}[\log|p|, a]b \tag{2.1.11}$$

which gives immediately the result we want.

In higher dimensions local multiplication operators satisfy a weaker condition. Let $\epsilon = p/|p|$ be the sign of a 'free' Dirac operator on a d-dimensional compact manifold; here $p = \sum \gamma^k p_k$ in the Clifford algebra. Assuming that X commutes with the Dirac gamma matrices, the commutator $[\epsilon, X]$ has then the principal symbol

$$[\epsilon, X] = -i(\frac{\gamma^j}{|p|} - \frac{p^j}{|p|^3})\partial_j X + ...$$

which is a PSDO of order -1. This raised to power $d+\delta$ for any positive δ is trace-class and therefore $[\epsilon, X] \in L_{d+}$, where L_p is the Schatten ideal of operators X such that $|X|^p$ is trace-class. We define the Lie algebras $\mathfrak{gl}_p$ as the Lie algebra of bounded operators for which $[\epsilon, X] \in L_p$. In particular, $\mathfrak{gl}_{res} = \mathfrak{gl}_2$. We have an infinite chain of Lie algebras $\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \mathfrak{gl}_3 \subset \dots \mathfrak{gl}_\infty$ with $\mathfrak{gl}_\infty$ the Lie algebra of operators with $[\epsilon, X]$ compact.

For example, in dimension $d = 3$, $X \in \mathfrak{gl}_{res}$ only when X is a constant. However, after subtracting a tail which behaves like $1/|p|$ for large momenta, a local multiplication operator X becomes a nonlocal operator which belongs to $\mathfrak{gl}_{res}$. For example, when $d = 3$ an a Riemannian manifold we can set

$$\tilde{X} = X + \frac{i}{4|p|^2}[\gamma^k, \gamma^l]p_k\partial_l X \tag{2.1.12}$$

and by a direct computation, using the defining property of Dirac matrices

$$\gamma_k \gamma_l + \gamma_l \gamma_k = 2\delta_{kl},$$

one sees that in the commutator $[\epsilon, \tilde{X}]$ all the terms which are of order $1/|p|$ cancel and the commutator is an operator of order -2. But in three dimensions such an operator is Hilbert-Schmidt and thus $\tilde{X} \in \mathfrak{gl}_{res}$.

2.2 A (bundle) gerbe and the Dixmier-Douady class from Hamiltonian anomalies

In this section we want to describe a more geometric approach to the problem of construction of the family of Fock spaces parametrized by external vector potentials and the action of the gauge group. Actually, the method here is very general and applies as well to the case of an external metric field or any other interactions for that matter. In order to keep the discussion as simple as possible we shall restrict to the case of vector potentials.

The fermionic Fock spaces parametrized by Yang-Mills potentials form a vector bundle $\mathcal{F}$ over the space $\mathcal{A}$. In the case of chiral massless fermions there are subtleties in defining this bundle. The difficulty is related to the fact that the splitting of the one particle fermionic Hilbert space H to positive and negative energies is not a continuous function of the external field. One can easily construct paths in the space of external fields such that at some point on the path a positive energy state dives into the negative energy space (or vice versa). These points are obviously discontinuities in the definition of the space of negative energy states and therefore the fermionic vacua do not form of smooth vector bundle over the space of external fields. This problem does not arise if we have massive fermions in the temporal gauge $A_0 = 0$. In that case there is a mass gap $[-m, m]$ in the spectrum of the Dirac hamiltonians and the polarization to positive and negative energy subspaces is indeed continuous.

If λ is a real number not in the spectrum of the hamiltonian then one can define a bundle of fermionic Fock spaces $\mathcal{F}'_{A,\lambda}$ over the set U_λ of external fields A, $\lambda \notin Spec(D_A)$. The vacuum in $\mathcal{F}'_{A,\lambda}$ is defined by the polarization of the one-particle space to positive and negative spectrum of the operator $D_A - \lambda$. It turns out that the Fock spaces $\mathcal{F}'_{A,\lambda}$ and $\mathcal{F}'_{A,\lambda'}$ are naturally isomorphic up to a phase. The phase is related to the arbitrariness in filling the Dirac sea between vacuum levels λ, λ'. Such a filling is given corresponds (because of the anticommutation relations) to an exterior product $v_1 \wedge v_2 \wedge \dots v_m$ of a complete orthonormal set of eigenvectors $D_A v_i = \lambda_i v_i$ with $\lambda < \lambda_i < \lambda'$. A rotation of the eigenvector basis gives a multiplication of the exterior product by the determinant of the rotation. Thus there is a well-defined complex line $DET_{\lambda\lambda'}(A)$ for each $A \in U_\lambda \cap U_{\lambda'} = U_{\lambda\lambda'}$ and

$$\mathcal{F}'_{A,\lambda'} = \mathcal{F}'_{A,\lambda} \otimes DET_{\lambda\lambda'}(A) \qquad (2.2.1)$$

over the intersection set. We set $DET_{\lambda'\lambda} = DET^{-1}_{\lambda\lambda'}$ for $\lambda < \lambda'$. Note that from these definitions follows immediately that the line $DET_{\lambda\lambda''}$ can be naturally identified as $DET_{\lambda\lambda'} \otimes DET_{\lambda'\lambda''}$, i.e., the local line bundles $DET_{\lambda\lambda'}$ form a cocycle over the open cover $\{U_\lambda\}$ of $\mathcal{A}$,

$$DET_{\lambda\lambda'} \otimes DET_{\lambda'\lambda''} = DET_{\lambda\lambda''}. \tag{2.2.2}$$

The complex line bundles $DET_{\lambda\lambda'}$ have a natural hermitean structure since they are defined using exterior powers of finite-rank subbundles in a Hilbert space and therefore we also have well-defined circle bundles by restriction to $S^1 \subset \mathbb{C}$.

If we choose a refinement of the open cover $\{U_\lambda\}$ such that the sets and their intersections are contractible we can choose local sections $\psi_{\lambda\lambda'}$ of the circle bundles. Because of the cocycle relation (2.2.2) we have

$$\psi_{\lambda\lambda'} \otimes \psi_{\lambda'\lambda''} \otimes \psi_{\lambda''\lambda} = f_{\lambda\lambda'\lambda''}\mathbf{1}$$

for S^1 valued functions $f_{\lambda\lambda'\lambda''}$ which satisfy the cocycle relation

$$f_{\alpha\beta\gamma} f^{-1}_{\alpha\beta\delta} f_{\alpha\gamma\delta} f^{-1}_{\beta\gamma\delta} = 1. \tag{2.2.3}$$

The curvature of the local line bundles $DET_{\lambda\lambda'}$ is defined as

$$\theta_{\lambda\lambda'} = \sum_i (d\psi_i, d\psi_i)$$

where the ψ_i's form an orthonormal basis for the spectral subspace $E_{\lambda\lambda'}$ corresponding to the spectral interval (λ, λ'). By construction, on triple intersections we have

$$\theta_{\lambda\lambda'} + \theta_{\lambda'\lambda''} + \theta_{\lambda''\lambda} = 0.$$

Choosing a partition of unity $\{\rho_\lambda\}$ for the open cover $U_{\lambda\lambda'}$ we can define forms

$$B_\lambda = \sum \rho_{\lambda'} \theta_{\lambda'\lambda}$$

on U_λ. Finally, the we have a form $H/(2\pi)^2$ representing a class in $H^3(\mathcal{A}, \mathbb{Z})$ (which is trivial for the affine space $\mathcal{A}$!) as

$$H = dB_\lambda,$$

and these agree on intersections. Although for the case of $\mathcal{A}$ the 3-form has to be exact in cohomology, we can pass to the quotient $\mathcal{A}/\mathcal{G}_0$ where it generically is nontrivial (depending on the gauge group G, its representation, and the dimensionality of M). The line bundles descend to $\mathcal{A} \to \mathcal{A}/\mathcal{G}_0$ since the Dirac operator transforms covariantly in gauge transformations and the line bundles DET where defined using spectral data.

The data $(H, B_\lambda, a_{\lambda\lambda'}, f_{\lambda\lambda'\lambda''})$ define a Deligne 2-cocycle on the base; here $a_{\lambda\lambda'}$ is a local potential for the curvature $\theta_{\lambda\lambda'} = da_{\lambda\lambda'}$ which can be defined as $a_{\lambda\lambda'} = \sum_i (\psi_i, d\psi_i)$. Then

$$a_{\lambda\lambda'} + a_{\lambda'\lambda''} + a_{\lambda''\lambda} = f^{-1}_{\lambda\lambda'\lambda''} df_{\lambda\lambda'\lambda''}.$$

The Deligne cocycles are essentially defined as the cocycles in a de Rham - Čech double complex with an important modification: Instead of real valued Čech cocycles on the degree zero de Rham level we have circle valued functions f with the *logarithmic* de Rham differential $f \mapsto f^{-1}df$.

In order to compensate the dependence on λ in the definition of the Fock spaces we search for a family of complex line bundles DET_λ over the open sets U_λ such that

$$DET_{\lambda'} = DET_{\lambda'\lambda} \otimes DET_\lambda \tag{2.2.4}$$

over $U_{\lambda\lambda'}$. Obviously, the cocycle property of the of the line bundles $DET_{\lambda\lambda'}$ is a necessary condition for the existence of the family of bundles DET_λ. It is not very hard to prove that this is also a sufficient condition. This follows also form the general theory of bundle gerbes [M. Murray, J. London Math. Soc. (2) 54 (1996), no. 2, 403 - 416] since $\mathcal{A}$ is topologically trivial.

We define the tensor product

$$\mathcal{F}_\lambda(A) = \mathcal{F}'_\lambda(A) \otimes DET_\lambda(A). \tag{2.2.5}$$

Using (2.2.1) and (2.2.4) we observe that the right-hand side is independent from λ and one has a well-defined bundle $\mathcal{F}$ of Fock spaces over all of $\mathcal{A}$.

Next one can ask what is the action of the gauge group in $\mathcal{F}$. The gauge action in U_λ lifts naturally to $\mathcal{F}'$. Thus the only problem is to construct a lift of the action on the base to the total space of DET_λ. Note that the determinant bundle here is a bundle over external fields in *odd dimension,* and therefore one would expect that it is trivial (curvature equal to zero) on the basis of families index theorem. However, it turns out that the relevant determinant bundle actually comes from a determinant bundle in even dimensions. Instead of single vector potentials we must study paths in $\mathcal{A}$, thus the extra dimension. The relevant index theorem is then the APS theorem for even dimensional manifolds with a boundary; physically, the boundary can be interpreted as the union of the space at the present time and in the infinite past, [A. Carey, J. Mickelsson, M. Murray Comm. Math. Phys. 183, no. 3, 707- 722(1997)].

We recall some facts about lifting a group action on the base space X of a complex line bundle to the total space E. Let ω be the curvature 2-form of the line bundle. It is integral in the sense that $\int \omega$ over any cycle is $2\pi\times$ an integer. Let G be a group acting smoothly on X. Then there is an extension $\hat{G}$ which acts on E and covers the G action on X. The fiber of $\hat{G} \to G$ is equal to $Map(X, S^1)$. As a vector space, the Lie algebra of the extension is $\mathfrak{g} \oplus Map(X, i\mathbb{R})$. The commutators are defined as

$$[(a,\alpha),(b,\beta)] = ([a,b], \omega(a,b) + \mathcal{L}_a\beta - \mathcal{L}_b\alpha) \tag{2.2.6}$$

where $a, b \in \mathfrak{g}$ and $\alpha, \underline{:}X \to i\mathbb{R}$. The vector fields generated by the G action on X are denoted by the same symbols as the Lie algebra elements a, b; thus $\omega(a,b)$ is the function on X obtained by evaluating the 2-form $\leq$ along the vector fields a, b. The Jacobi identity

$$\omega([a,b],c) + \mathcal{L}_a\omega(b,c) + \text{cyclic permutations} = 0$$

for the Lie algebra extension $\hat{\mathfrak{g}}$ follows from $d\omega = 0$.

What we need is a formula for the curvature of the line bundles DET_λ along gauge directions. Not surprisingly, this is given by a reduction from a secondary characteristic class. Recall that in even space-time dimensions the Chern class of the determinant line bundle is obtained by starting from an appropriate characteristic class (the class appearing in the index formula of Dirac operators) in two higher dimensions and then integrating over the space-time manifold; this leaves a closed integral differential form of degree two on the parameter space of the Dirac operators, [Atiyah-Singer 1984]. In the odd dimensional case here one starts from the Atiyah-Patodi-Singer index formula on a manifold with a boundary. The formula contains two pieces on the right-hand side. The first is an integral of a local differential polynomial (the same as in the case without boundary) and the second is the so-called eta-invariant which contains nonlocal information about the spectrum of the boundary Dirac operator. The essential property of the eta-invariant is that it is gauge invariant. For that reason it does not give a contribution to the curvature of the determinant bundle along gauge orbits. Everything comes from the local differential polynomial; the non- gauge invariant piece of the latter comes from the boundary and is equal to a secondary characteristic class. In simple situations this is just a Chern-Simons form.

Integrating the Chern-Simons form in $2n+3$ dimensions over the $2n+1$ dimensional physical space gives a 2-form along gauge orbits.

For example, when $\dim M = 1$, starting from the Chern-Simons form $\frac{1}{8\pi^2}\mathrm{tr}(AdA + \frac{2}{3}A^3)$ we get

(2.2.7) $$\omega_A(X,Y) = \frac{1}{4\pi}\int_{S^1} \mathrm{tr}\, A_\phi [X,Y],$$

the curvature at the point A in the directions of infinitesimal gauge transformations X, Y. (Note the normalization factor 2π relating the Chern class to the curvature formula.) This is not quite the central term of an affine Kac-Moody algebra, but it is equivalent to it (in the cohomology with coefficients in $\mathrm{Map}(\mathcal{A}, \mathbb{C})$). In other words, there is a 1-form θ along gauge orbits in $\mathcal{A}$ such that $d\theta = \omega - c$, where

(2.2.8) $$c(X,Y) = \frac{i}{2\pi}\int \mathrm{tr} X\partial_\phi Y$$

is the central term of the Kac-Moody algebra, considered as a closed constant coefficient 2-form on the gauge orbits. There is a simple explicit expression for θ,

$$\theta_A(X) = \frac{i}{4\pi}\int \mathrm{tr} AX.$$

When $\dim M = 3$ the curvature (or equivalently, the Schwinger term) is obtained from the five dimensional Chern-Simons form

$$CS_5(A) = \frac{i}{24\pi^3}\mathrm{tr}(A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5).$$

By the same procedure as in the one dimensional case we obtain

(2.2.9)
$$\omega_A(X,Y) = \frac{i}{4\pi^2}\int \mathrm{tr}\left((AdA + dA\,A + A^3)[X,Y] + XdA\,YA - YdA\,XA\right).$$

This differs from the Mickelsson-Faddeev-Shatashvili cocycle [J. Mickelsson, Comm. Math. Phys. 97 (1985), no. 3, 361- 370; L. Faddeev, S. Shatashvili, Teoret. Mat. Fiz. 60 (1984), no. 2, 206- 217

$$\omega'_A(X,Y) = \frac{i}{24\pi^2}\int \mathrm{tr} A(dXdY - dYdX)$$

by the coboundary of

$$\frac{-i}{24\pi^2}\int \mathrm{tr}(AdA + dA\,A + A^3)X.$$

Now we have a projective action of the gauge group $\mathcal{G}$ on the Fock bundle $\mathcal{F}$ over $\mathcal{A}$. Because of the projective phases this can be pushed-forward to $\mathcal{G}/\mathcal{G}_0$ only as a projective bundle $P\mathcal{F}$. Any infinite-rank complex projective bundle is determined up to an isomorphism by its *Dixmier-Douady class* in the third integral cohomology of the base. So what is the D-D class in our case? It can again be extracted from the families index theorem. That is, it is given by the formula (1.3.9). One needs to fix a connection in a vector bundle E over the space $M \times \mathcal{A}/\mathcal{G}_0$. This has been done in [Atiyah-Singer, 1984] ; the only problem is that the formula involves the Green's function of the Dirac Laplacian and therefore is nonlocal.

However, in a simple situation one can derive a very explicit formula. This is the case when $M = S^1$. Now the vector potentials can be written as $A = f^{-1}df$ for some smooth function $f : [0, 2\pi] \to G$. Modulo based gauge transformations $g, g(0) = 1 = g(2\pi)$, the configurations are parametrized by elements $f(2\pi) \in G = \mathcal{A}/\mathcal{G}_0$. Let us denote by $\mathcal{P}G$ this space of paths f. A family of Dirac operators on the circle S^1 can be defined starting from the trivial bundle vector bundle over $S^1 \times \mathcal{P}G$ with fiber a complex vector space V carrying an unitary representation of G. We define a vector bundle E over $S^1 \times G$ by the equivalence relation

$$(x, f, v) \equiv (x, fg, g(x)^{-1}v)$$

where $g \in \mathcal{G}_0$. A connection in this bundle is given as a 1-form on $S^1 \times \mathcal{P}G$ with values in $\mathfrak{g}$,

$$\omega = f^{-1}df - \alpha(x)f(x)^{-1}[df(2\pi)f(2\pi)^{-1}]f(x)$$

where α is ant smooth function on the interval $[0, 2\pi]$ such that $\alpha(0) = 0, \alpha(2\pi) = 1$, and the derivatives are periodic at the end points. The only role of α is to make the form ω periodic. By choosing a local section $a \mapsto f_a$ from G to $\mathcal{P}G$ one obtains a local connection 1-form on the base $S^1 \times G$ of E. This form splits to components $\omega = \omega^{(1,0)} + \omega^{(0,1)}$ along S^1 and G. The curvature F is easily computed and is

$$\begin{aligned} F^{(1,1)} &= -d\alpha f(x)^{-1}[df(2\pi)f(2\pi)^{-1}]f(x) \\ F^{(0,2)} &= (\alpha^2 - \alpha)f(x)^{-1}[df(2\pi)f(2\pi)^{-1}]^2 f(x) \end{aligned}$$

and $F^{(2,0)} = 0$ for dimensional reasons. By the families index theorem, the 3-cohomology part of the odd Chern character is given by a fiber integration over S^1,

$$\begin{aligned} H = c_3 &= \frac{1}{8\pi^2}\int_{S^1} \mathrm{tr}\,(F^2)^{(1,3)} = -\frac{1}{4\pi^2}\int_{S^1} d\alpha(\alpha^2-\alpha)f(x)^{-1}[df(2\pi)f(2\pi)^{-1}]^3 f(x) \\ &= \frac{1}{24\pi^2}\mathrm{tr}\,[df(2\pi)f(2\pi)^{-1}]^3 \end{aligned}$$

which in the case of $G = SU(n)$ in the fundamental representation is the canonical generator of $H^3(G, \mathbb{Z})$. For any simple Lie group in an arbitrary representation it is an integral multiple of the generator.

The index theory computation of the Dixmier-Douady class H does not take into account possible torsion information. For that problem one has to use more refined methods. See the monograph [J. Mickelsson, Current groups and algebras, 1989] for the case $G = SU(2)$ when $\dim M = 3$ and [A. Carey, J.M., M. Murray, Rev. Math. Phys. 12 (2000), no. 1, 65- 90] for a more general case.