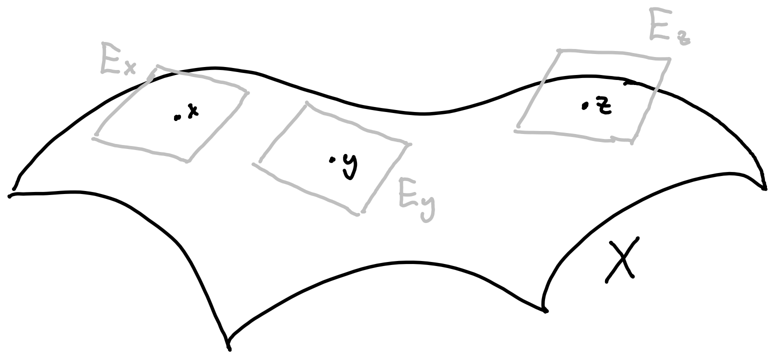
Vector bundles and characteristic classes.



A vector bundle over a manifold consists of a continuously varying family of vector spaces at each point.

Def. A vector bundle is a manifold $\pi : E \to X$ such that

(1) The fibres $E_x := \pi^{-1}(x)$ have the structure of a vector space.

(2) For each $x \in X$, there exists an open neighborhood U of x together with

a diffeomorphism

$$\Phi : E|_U := \pi^{-1}(U) \longrightarrow U \times \mathbb{K}^m$$

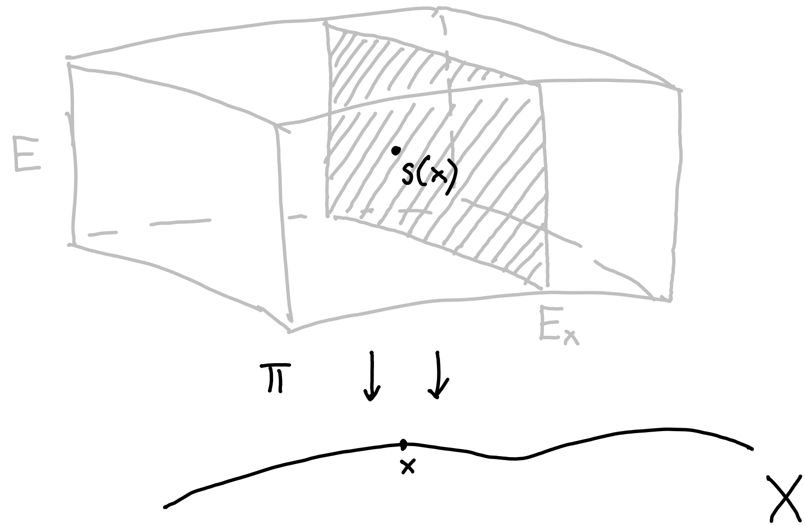
that is an isomorphism of vector spaces when restricted to each fiber.

Here $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and m is the dimension or the rank of E (it is necessarily locally constant, i.e. constant on each component, and sometimes required to be constant globally).

Def. A section of a vector bundle E is a smooth map $u : X \to E$ such that $\pi \circ u = \mathrm{id}_X$.

In other words, u assigns to each point a vector over this point.

Other way of drawing things:



<u>Def</u>. A vector bundle is trivial if it has a <u>global</u> trivialisation.

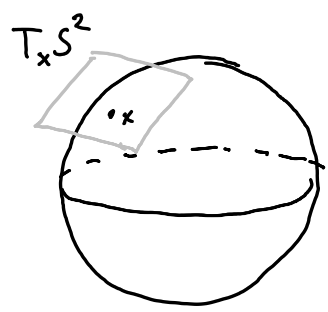
In this case, we can find $m = rk(E)$ linearly independent, non-vanishing sections. This can be used to test triviality.

Ex. The tangent bundle of a manifold consists of the collection of its tangent vectors.

$$TX = \{(x,v) \mid x \in X,\ v \in T_x X\}.$$

Derived from this are several tensor bundles, such as $\mathrm{End}(TM)$, $\Lambda^k TM$, ...

Ex. (a) The tangent bundle of S^2.



This is not globally trivial, as by the Poincaré–Hopf theorem, each (non-degenerate) vector field has at least $\chi(S^2) = 2$ zeroes. We can argue similarly for all even-dimensional spheres. For odddimensional spheres, $\chi(S^{2k+1})=0$, so this argument does not work. Indeed, the spheres S^1, S^3, S^7 have trivial tangent bundle (but only those).

Let $\Gamma(E)$ denote the space of sections of a vector bundle.

<u>Def</u>. A <u>connection</u> is a bilinear map

$$\nabla : \Gamma(TX) \times \Gamma(E) \longrightarrow \Gamma(E)$$

such that for $V \in \Gamma(TX)$, $\varphi \in \Gamma(E)$, $f \in C^\infty(X)$

(1) $\nabla_{fV} \varphi = f \nabla_V \varphi$

(2) $\nabla_V f\varphi = \partial_V f \cdot \varphi + f \nabla_V \varphi$.

If E carries a fibrewise scalar product (or a pos. def. Hermitean form in the complex case), we require additionally

$$\partial_V h(\varphi, \psi) = h(\nabla_V \varphi, \psi) + h(\varphi, \nabla_V \psi).$$

If X is a Riemannian manifold, it is a fact that there exists a unique connection ∇ on the tangent bundle such that (3) holds, and we additionally have

$$(4) \quad \nabla_V W - \nabla_W V = [V, W] \quad \text{(Lie bracket)}$$

This is called the <u>Levi-Civita connection</u>.

For functions

$$\varphi : X \longrightarrow \mathbb{K}^m,$$

we have Schwarz's theorem

$$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\varphi\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\varphi\right)$$

with respect to local coordinates.

For functions with values in vector bundles, i.e. sections, this is not true. The failure for this to hold is the <u>curvature</u>.

Def. The *curvature* of a connection ∇ is the trilinear form

$$F = F^{\nabla} : \Gamma(TX) \times \Gamma(TX) \times \Gamma(E) \to \Gamma(E)$$

given by

$$F(V,W)\varphi = \nabla_V \nabla_W \varphi - \nabla_W \nabla_V \varphi - \nabla_{[V,W]}\varphi.$$

Lemma. The curvature is actually a tensor, i.e.

$$F(fV,W)\varphi = F(V,fW)\varphi = F(V,W)f\varphi$$
$$= f\,F(V,W)\varphi.$$

Moreover,

$$F(V,W)\varphi = -F(W,V)\varphi,$$

hence $F \in \Gamma(\Lambda^2 TX \otimes \mathrm{End}(E))$.

In particular, we have the Riemann curvature tensor R, which is the curvature tensor of the Levi-Civita connection.

In fact, R has the additional symmetries

$$\langle R(V,W)Y, Z\rangle = -\langle R(V,W)Z, Y\rangle$$
$$= \langle R(Y,Z)V, W\rangle$$

and the Bianchi identity

$$R(V,W)Y + R(Y,V)W + R(W,Y)V = 0.$$

Ex. The curvature tensor on S^n is

$$R(V,W)Y = \langle V,Y\rangle W - \langle W,Y\rangle V$$

Characteristic classes

Of course, if a vector bundle is trivial, it possesses a flat connection, i.e. one where $F \equiv 0$. Conversely curvature can be used to measure non-triviality of a vector bundle.

As seen before, the curvature tensor of a connection can be seen as an element of the space

$$\Omega^2(X, \mathrm{End}(E)) = \Gamma(\Lambda^2 T^*X \otimes \mathrm{End}(E))$$

of $\mathrm{End}(E)$-valued 2-forms.

Given a local trivialization of E, we can see this also as an $m \times m$ matrix of $\mathbb{K}$-valued 2-forms, denoted by

$$\Omega \in \Omega^2(X, \mathfrak{g}) \subseteq \mathrm{Mat}_{m\times m}(\Omega^2(X)).$$

<u>Def</u>. For $G \subseteq GL(m)$ a subgroup, with Lie algebra $\mathfrak{g} \subseteq \mathrm{Mat}_{m\times m}(\mathbb{K})$, a polynomial

$$P : \mathfrak{g} \longrightarrow \mathbb{K}$$

is <u>G-invariant</u>, if

$$P(QAQ^{-1}) = P(A)$$

for all $Q \in G$.

Ex. For $G = GL(m, \mathbb{K})$, then we have $P = \mathrm{tr}, \det$, and more generally, $P =$ any symmetric polynomial of the eigenvalues.
If $\mathbb{K} = \mathbb{R}$ and $G = SO(m)$ with m even, we additionally have $P = \mathrm{pf}$, the Pfaffian.

Fact: For $G = GL(m)$ or $O(m)$, the algebra of homogeneous polynomials is generated by $\mathrm{tr}(A^k)$, $k \in \mathbb{N}$.

For such a polynomial, we can form

$$P(\Omega) \in \Omega(X, \mathbb{K}),$$

which is a $2k$-form if P is homogenous of degree k. It is indepent of the choice of trivialization, by invariance of P.

Lemma. $P(\Omega)$ is closed.

Proof. We may assume that

$$P(\Omega) = tr(\Omega^k).$$

$$dP(\Omega) = tr(d\Omega^k) = \sum_{j=1}^{k} tr(\Omega^{j-1} d\Omega\, \Omega^{k-j})$$

$$= k\, tr(\Omega^{k-1} d\Omega).$$

Now the Bianchi identity states that

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega,$$

where with respect to the local trivialization used to define Ω, $\omega \in \Omega^1(X, \mathfrak{g})$ is given by

$$\nabla_v \varphi = \partial_v \varphi + \omega(v)\varphi.$$

Now

$$tr(\Omega^{k-1} d\Omega) = tr(\Omega^k \wedge \omega - \Omega^{k-1} \wedge \omega \wedge \Omega)$$

$$= tr([\Omega^k, \omega])$$

$$= 0$$

$\square$

Hence $P(\Omega)$ defines a class in de-Rham cohomology.

For a map $f: Y \to X$, we can construct the pullback bundle f^*E over Y, and a connection ∇ on E induces a connection $f^*\nabla$ on f^*E.
We can also pullback trivialisations, and in this sense, we have the formula

$$\Omega^{f^*\nabla} = f^*\Omega \in \Omega^2(Y, \mathfrak{g}).$$

This implies

$$P(\Omega^{f^*\nabla}) = f^* P(\Omega),$$

independently of any trivialisation.
From this follows

<u>Thm.</u> $P(\Omega)$ is in fact independent from the connection used to define it.

Proof. Let ∇^0, ∇^1 be two connections. Lift the vector bundle E to $X \times \mathbb{R}$, and consider the connection

$$\nabla^s = (1-s)\nabla^0 + s\nabla^1$$

on this bundle. Then

$$P(\Omega^j) = \iota_j^* P(\Omega^{\cdot}) \qquad j = 0, 1$$

for $\iota_j : X \longrightarrow X \times \mathbb{R}$ being the inclusions at j. Since ι_0, ι_1 are homotopic, the classes $\iota_0^* P(\Omega^{\cdot})$, $\iota_1^* P(\Omega^{\cdot})$ define the same class in de-Rham cohomology. $\square$

Examples. (1) $P(A) = \det\left(I + \frac{1}{2\pi i} A\right)$, $G = GL(m, \mathbb{C})$, for complex vector bundles. These are called the Chern classes

$$c(E) = 1 + c_1(E) + c_2(E) + \dots$$

where

$$c_k(E) \in \Omega^{2k}(X).$$

(2) The Pontrjagin classes are derived from this. They are defined for a real bundle W by

$$p_k(W) = c_{2k}(W \otimes \mathbb{C}) \in \Omega^{4k}(X).$$

Note that in general, we have

$$c_k(\bar{E}) = (-1)^k c_k(E),$$

where $\bar{E}$ is the complex conjugate bundle. Hence, because for a real bundle W, we have

$$\overline{W \otimes \mathbb{C}} \cong W \otimes \mathbb{C},$$

hence all odd Chern classes vanish,

$$c_k(W \otimes \mathbb{C}) = -c_k(\overline{W \otimes \mathbb{C}}) = -c_k(W \otimes \mathbb{C}).$$

(3) For an oriented real vector bundle, the <u>Euler class</u> is

$$\mathrm{Eul}(W) = \mathrm{pf}(\Omega).$$

(4) The <u>Chern character</u> is defined by

$$\mathrm{ch}(E) = \mathrm{tr}\exp\left(\frac{\Omega}{2\pi i}\right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{tr}\left(\left(\frac{\Omega}{2\pi i}\right)^k\right)$$

Note that this is in fact only a finite sum.

(5) The $\hat{A}$-class is

$$\hat{A}(W) = \det{}^{1/2}\left(\frac{\frac{\Omega}{4\pi i}}{\sinh\left(\frac{\Omega}{4\pi i}\right)}\right).$$

This is meant as follows. The even function

$$f(x) = \frac{x/2}{\sinh(x/2)}$$

satisfies $f(0) = 1$, hence it has a unique even analytic squareroot

$$g(x) = 1 + \sum_{k=1}^{\infty} g_k x^{2k}$$

near zero. We then set

$$\hat{A}(W) = \det\left(I + \sum_{k=1}^{\infty} g_k \left(\frac{\Omega}{2\pi i}\right)^{2k}\right).$$

$$\in \Omega^{4*}(X).$$

Examples.

(1) On spheres, all Pontrjagin classes vanish, as they are stable,

$$p_k(W) = p_k(W \oplus \underline{\mathbb{R}}^m),$$

where $\underline{\mathbb{R}}^m$ denotes the trivial bundle. As spheres are embedded hypersurfaces, one has

$$TS^n \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}.$$

(2) For the Euler class, one has

$$Eu((S^{2k+1}) = 0 \qquad \int_{S^{2k}} Eu(S^{2k}) = 2.$$

Note that here, we need to choose an orientation in order to have these forms defined.

(3) After choosing an orientation and on S^2, there is a unique complex structure J defined by sending $Jv = w$, $Jw = -v$, if (v, w) is a positively oriented orthonormal basis. Considering TS^2 as a complex bundle this way, we have

$$\int_{S^2} c_1(TS^2_J) = 2 \qquad \text{(I think)}$$

In general, a complex vector bundle E of complex dimension k can be seen as a real oriented bundle via the complex structure, and in this sense, we have

$$c_k(E) = \mathrm{Eul}(E).$$

Chern and Pontrjagin classes are multiplicative in the sense that

$$c(E \oplus F) = c(E) \wedge c(F)$$
$$p(E \oplus F) = p(E) \wedge p(F).$$

Thm. S^{4k} cannot carry a complex structure.

Proof. Assume the converse. Then

$$TM_J \oplus \overline{TM_J} \cong TM \otimes \mathbb{C}$$

via

$$(v, w) \longmapsto \frac{1}{\sqrt{2}}(v \otimes 1 - Jv \otimes i) + \frac{1}{\sqrt{2}}(w \otimes 1 + Jw \otimes i).$$

Hence

$$0 = c(TM \otimes \mathbb{C}) = c(TM_J)\, c(\overline{TM_J}).$$

$$= \left(1 + c_{2k}(TM_J)\right)\left(1 + c_{2k}(TM_J)\right)$$

$$= 1 + 2c_{2k}(TM_J)$$

$$= 1 + 2\,\mathrm{Eul}(TM).$$

Hence

$$0 = 2\int_{S^{4k}} \mathrm{Eul}(TM) = 4,$$

a contradiction. $\square$