SHEAF COHOMOLOGY VIA ABELIAN CATEGORIES

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Abstract. One of the most powerful cohomology theories is sheaf cohomology, that is, cohomology with coefficients in a sheaf of modules on a space. It is defined using derived functors on abelian categories with enough injectives, in great generality, but can be computed in a number of concrete ways which are familiar from differential geometry. The aim of this talk to give an idea of the technical origins of sheaf cohomology and also to illustrate these concepts with down-to-earth examples. A basic familiarity with categories and functors will be assumed.

The general setting of sheaf cohomology was developed in the late 1950s, motivated in part by the desire to have a good definition of cohomology with coefficients in a sheaf for varieties or schemes. Since in the natural topology (the Zariski topology) these are neither Hausdorff nor locally contractible, the usual constructions like Čech or singular cohomology fail. Even though we may be interested in smooth or complex manifolds, the machinery of abelian categories allows us to define sheaf cohomology in this setting and it can then be shown to give the same results as other cohomology theories where the sheaf coefficients are taken to be coefficients appropriate to the other theories.

In fact Grothendieck defined abelian categories precisely so that the tools of homological algebra would be available and that cohomology functors could be defined without recourse to haphazard constructions. However, the older techniques of constructing cohomology are much more useful in practice, and can be approached from the abstract side as well. This illustrates how one might calculate cohomology in settings other than the motivating problem. viz. sheaf cohomology

1. Lightning review of sheaves

We fix a topological space X, such as a manifold. Let $Open(X)$ denote the category whose objects are open sets $U \subset X$ and arrows are inclusion maps $U \subset V$. Recall that a sheaf on X is a functor

$$Open(X)^{op} \to Set$$

and similarly a sheaf of abelian groups (resp. rings) is a functor $Open(X)^{op} \to Ab$ (resp. $Open(X)^{op} \to Ring$). A map between sheaves is a natural transformation, and so we get the category $Ab(X)$ of sheaves of abelian groups. If we fix a sheaf of rings $\mathcal{O}_X$, we can talk about $\mathcal{O}_X$-modules, which are sheaves of abelian groups with an $\mathcal{O}_X$-action. We thus have the category $\mathcal{O}_X Mod$ of $\mathcal{O}_X$-modules. (In fact $Ab(X)$ is the category of modules for the constant sheaf $\mathbb{Z}$, see below for the definition.)

This is rather abstract, and there is an equivalent way to see sheaves (followed, for instance, in Hirzebruch's book on algebraic topology). Recall that a map $S \to X$ of

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spaces is called a local homeomorphism (or étale) if every point $s \in S$ has an open neighbourhood which is mapped homeomorphically to an open neighbourhood of the image of s. A map between such things is just a map *over* X.

To get from an étale $p\colon S \to X$ map to a sheaf one takes the sheaf

$$U \mapsto \Gamma(U, S)$$

of local sections of p. The fibre at x is then the limit of $\Gamma(U, S)$ as U ranges over open sets $U \ni x$. This limit is called the *stalk* at x. We can go from a sheaf to an étale map by defining S, for a given sheaf, as the union of the stalks, appropriately topologised. There is an equivalence of categories between categories of sheaves and the category of étale maps with codomain X.

A sheaf of abelian groups, rings or modules corresponds to an étale space which has the structure of an abelian group, ring, module on its fibres, or equivalently is an abelian group object (etc.) in the category of étale spaces over X.

An easy example is the *constant sheaf with fibre* Y, where Y is a set, or abelian group etc., which is given as local sections of the projection map $X \times Y \to X$ (Y has the discrete topology. Thus it is given by locally constant functions $X \supset U \to Y$. Another example is given by the *sheaf* $\underline{k}$ *of germs if* k*-valued functions* for $k \in \{\mathbb{R}, \mathbb{C}\}$, where we can take continuous, smooth, analytic, holomorphic etc. as appropriate. This is defined as the sheaf of local sections of the projection $X \times k$ for k with the usual topology.

More generally, we might consider the sheaf of local sections of a vector bundle, such as the tangent bundle, cotangent bundle, or symmetric or exterior powers of these. If we have a k-vector bundle, then the sheaf of sections is a $\underline{k}$-module. Do note that the étale space associated with these sheaves looks very different to the vector bundle!

2. Abelian categories and derived functors

Abelian categories are categories for which the hom-sets are abelian groups, composition is a homomorphism of abelian groups, and for which the usual constructions of rings and modules 'just work'. In fact it is a theorem that every (small) abelian category is a full subcategory of the category of modules for some ring. It is safe for the current talk to assume we will be working with the category of sheaves of abelian groups on a fixed space, or with the category of $\mathcal{O}$-modules where $(X, \mathcal{O}_X)$ is a fixed ringed space. (More concretely, you can think of X as a manifold, smooth or complex, and $\mathcal{O}_X$ the sheaf of smooth or holomorphic functions on X). The category *Ab* of abelian groups or the category of R-modules can be recovered by taking our fixed space to be a point.

Definition 2.1. An abelian category A is a category *enriched* over *Ab*, with finite *biproducts* (=direct sum = direct product) such that

AB1 For every map $f\colon a \to b$ we have

$$0 \to \ker f \to a$$

and

$$b \to \operatorname{coker} f \to 0$$

(note that there is a zero object and a zero morphism between any two objects),

AB2 For every map $f\colon a \to b$ the canonical map $a/\ker f \to \operatorname{im} f$ is an isomorphism.

We have the usual machinery of (short) exact sequences and (co)chain complexes in abelian categories. We have the result that the opposite of an abelian category is again an abelian category.

There are several sorts of functors between abelian categories that are important, we shall focus on just two of them. All functors between abelian categories are assumed to be *additive* functors, that is they respect the group structure on hom-sets.

Definition 2.2. A functor $F\colon A \to B$ between abelian categories is called *left exact* if for all short exact sequences

$$0 \to a \to b \to c \to 0$$

in A we have an exact sequence

$$0 \to F(a) \to F(b) \to F(c).$$

F is called *exact* if we can extend the above exact sequence to the right by 0. Left exact functors preserve direct sums.

The most important example for us is the functor $Hom(-,a)\colon A^{op} \to Ab$, which is left exact. (Note that since A^{op} is abelian for A abelian, we can talk about (left) exact *contravariant* functors, i.e. functors out of A^{op} without difficulty.)

An important class of objects in abelian categories are called *injective*. They are formally dual to projective objects, of which projective modules are an example. In Ab the injective objects are the divisible groups. We give a definition which is equivalent to the more formal definition.

Definition 2.3. An object I of an abelian category A is called *injective* if the left exact functor $Hom(-,I)\colon A^{op} \to Ab$ is exact. Equivalently, for every monomorphism $I \to a$ there is a retract $a \to I$.

The importance of injective objects is that they allow us to define cohomology as a *derived functor*. We will not give the formal definition of derived functors, only show a construction, but they are defined via a universal property. We can define the (right) derived functor when an abelian category has *enough injectives*.

Definition 2.4. An abelian category A has *enough injectives* if for every object a there is a monomorphism $a \to I$ for some injective object I.

Importantly, the category $RMod$ of R-modules for a fixed ring R has enough injectives: embed the module M into

$$\prod_{Ab(M,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$$

where $Ab(M,\mathbb{Q}/\mathbb{Z})$ denotes homomorphisms out of the abelian group underlying M. This inherits an R-action making it an R-module.

From property we can find for every object a of A an *injective resolution*, a long exact sequence

$$0 \to a \to I^0 \to I^1 \to I^2 \to \cdots (= 0 \to a \to I^\bullet)$$

Assume we have a left exact additive functor $A \to Ab$. We then define, subject to proving the definition is (up-to-isomorphism) independent of choices, the right derived functors $R^iF\colon A \to Ab$ as

$$R^iF(a) = H^i(F(I^\bullet)), \quad i \geq 0$$

where we can take cohomology because $F(I^\bullet)$ is a cochain complex.

Theorem 2.5. *Given an abelian category A with enough injectives and a left exact additive functor $F\colon A \to Ab$ the right derived functors R^iF are additive, R^0F is naturally isomorphic to F and for every short exact sequence $0 \to a \to b \to c \to 0$ there is a long exact sequence*

$$0 \to F(a) \to F(b) \to F(c) \to R^1F(a) \to R^1F(b) \to R^1F(c) \to R^2F(a) \to \cdots$$

in Ab.

This result holds true for left exact additive functors $A \to B$, but we only need the above for what we are going to do.

3. Abelian categories of sheaves

Proposition 3.1. *The categories $Ab(X)$ and $\mathcal{O}_X Mod$ are abelian.*

Here it is easiest to describe kernels and cokernels using étale maps: for the kernel of $f\colon a \to b$ take the subspace of a mapping to the zero section of $b \to X$ (this exists!). For the cokernel take the coequaliser of f and the zero map. (In technical terms, we need to sheafify the presheaf given by taking the naive cokernel of the sheaf defined as a functor.)

In particular, one can check that taking global sections preserves kernels, but not cokernels. For the purposes of constructing sheaf cohomology, the only left exact additive functor we are interested in is this global sections functor $\Gamma\colon \mathcal{O}_X Mod \to Ab$.

The key result that Grothendieck proved which enabled the definition of sheaf cohomology as a derived functor is this

Theorem 3.2. *The category $\mathcal{O}_X Mod$ has enough injectives.*

The proof relies on knowing the analogous result for modules, and essentially consists of embedding each stalk of a sheaf of modules into an injective module I_x, defining a space over X to be the product $\prod_{x\in X} I_X$ of all of these stalks, then defining the needed injective sheaf as the sheaf of discontinuous sections of the projection map.

From the abstract considerations in the first section, we get the right derived functors of Γ which we call cohomology: $H^i(X, M) := R^i\Gamma(M)$ for some $\mathcal{O}_X$-module M.

As you may have guessed, calculating H^i from the injective resolution guaranteed by the construction of an injective sheaf is not practical. Thus we turn to other sorts of resolutions.

Definition 3.3. A sheaf a on X is called *acyclic* if $H^i(X, a) = 0$ for all $i > 0$.

We define an acyclic resolution the same way as an injective resolution.

Theorem 3.4. *Let f be a sheaf on X and let $0 \to f \to a^\bullet$ be an acyclic resolution. Then $H^i(X, f) \simeq H^i(\Gamma(X, a^\bullet))$*

So we need to come up with some examples of acyclic resolutions. We do this via some supplementary definitions.

Definition 3.5. A sheaf f on X is called *soft* if for every closed set $K \subset X$ the map $\Gamma(X, f) \to \Gamma(K, f)$ given by restriction is surjective. A sheaf is called *fine* if for any locally finite covering of X there is a partition of unity subordinate to that covering.

Lemma 3.6. *If a sheaf of (unital) rings $\mathcal{O}$ is soft, then any $\mathcal{O}$-module is fine.*

Proposition 3.7. *Soft and fine sheaves are acyclic.*

Hence we can calculate sheaf cohomology using fine sheaves. The most important examples we need for geometry are

- The sheaf of smooth differential forms $U \mapsto \Omega^p(U)$, $p \geq 0$ on a smooth manifold is fine.
- Let E be a holomorphic vector bundle over a complex manifold. The sheaf of (p, q)-differential forms with values in E is fine.

This means that we can calculate cohomology with values in $\mathbb{R}$ and E by the de Rham complex and the Dolbeault complex respectively.

In other settings one wants to find acyclic resolutions appropriate to the objects one is working with.

Here is another application that we might be interested in: the sheaf of germs of continuous $\underline{\mathbb{R}}$-valued functions is soft. Now recall we have an exact sequence of sheaves

$$0 \to \mathbb{Z} \to \underline{\mathbb{R}} \to \underline{U(1)} \to 0$$

By the definition of sheaf cohomology as a derived functor we get a long exact sequence

$$\cdots \to H^i(X, \mathbb{Z}) \to H^i(X, \underline{\mathbb{R}}) \to H^i(X, \underline{U(1)}) \to H^{i+1}(X, \mathbb{Z}) \to \cdots$$

But since $\underline{\mathbb{R}}$ is a soft sheaf it is acyclic, and hence we get isomorphisms

$$H^i(X, \underline{U(1)}) \simeq H^{i+1}(X, \mathbb{Z})$$