Seiberg-Witten theory 101. 2018. 3.

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Ref. Salamon "Spin geometry and Seiberg-Witten invariants"

§1. What is SW invariant?

The SW invariant is an invariant of oriented closed smooth 4-manifolds.

$$SW(X,-) : \{\text{spin}^c \text{ structures on } X\}/_{\text{iso}} \to \mathbb{Z}$$

$$\mathfrak{s} \longmapsto SW(X,\mathfrak{s})$$

- Spin^c str.

$$\begin{array}{ccc} \mathrm{Spin}^c(4) := \dfrac{Sp(1)\times Sp(1)\times U(1)}{\pm 1} & \ni & [q_+, q_-, \lambda] \\ \downarrow & & \downarrow \\ SO(4) & = & (v \mapsto q_- \cdot v \cdot q_+^{-1}) \end{array}$$

$Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$

$v \in \mathbb{R}^4 = \mathbb{H}$

Def.

A <u>spin^c str.</u> on X^4 is a pair (P, ψ),

where

- $\mathrm{Spin}^c(4) \to P \to X$ is a principal $\mathrm{Spin}^c(4)$-b'dle.
- $\psi : P \times_{\mathrm{Spin}^c(4)} SO(4) \xrightarrow{\cong} Fr_X$ ← frame b'dle of X.

  : an iso as $SO(4)$-b'dle.

Recipe to define $SW(X, \mathfrak{s})$:

Step 1. Fix a Riemannian metric (and a "perturbation").

Step 2. Then we can write down the SW equations (← a non-linear PDE).

Step 3. Define

$$SW(X, \mathfrak{s}) := \# \overbrace{\{\text{solutions to SW eq}\} / \underbrace{\text{symmetry}}_{\text{gauge group}}}^{\text{called the moduli space}}$$

$SW(X, \mathfrak{s}) \in \mathbb{Z}$

Step 4. Show that $SW(X, \mathfrak{s})$ is indep. of choice of metric (& perturbation).

This recipe is quite similar to Donaldson invariant & Gromov-Witten inv.

Biggest feature of SW theory

... The moduli space is cpt!

Rem. To define SW inv., we need a topological assumption : $b^+(X) \geq 2$.

$$Q_X : H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(\alpha, \beta) \longmapsto \langle \alpha \cup \beta, [X] \rangle$$

the intersection form of X

Q_X is a symmetric, non-deg. bilinear form.

$b^+(X) :=$ dim. of a maximal positive definite subspace of $H^2(X;\mathbb{Z})$ w.r.t. Q_X.

$b^-(X) :=$ ―――――――― negative ――――――――.

e.g.

$$Q_{\mathbb{C}P^2} = (1) \qquad b^+(\mathbb{C}P^2) = 1,$$

$$Q_{-\mathbb{C}P^2} = (-1). \qquad b^-(-\mathbb{C}P^2) = -1. \quad ⌟$$

§2. Applications

◎ Obstruction to connected sum decomposition

X : oriented closed smooth 4-mfd.

Thm

Assume that $b^+(X) \geq 2$ and

$SW(X, \mathfrak{s}) \neq 0$ for some spin^c str. $\mathfrak{s}$ on X.

Then,

$\nexists X_i$: an ori. closed smooth 4-mfds $(i = 1, 2)$

s.t. $\begin{cases} X = X_1 \# X_2, \\ b^+(X_i) \geq 1. \end{cases}$

e.g. Taubes showed that if X is a symplectic 4-mfd, X admits a spin^c str. $\mathfrak{s}_{can}$ s.t. $SW(X, \mathfrak{s}_{can}) = 1$.

So X does not admit a decomposition

$$X = X_1 \# X_2 \text{ w/ } b^+(X_i) \geq 1.$$

For example, $\overset{n}{\#} \mathbb{CP}^2$ $(n \geq 2)$ doesn't admit a symplectic str.! (Note $\mathbb{CP}^2$ do admit!)

$\square$

- Distinguish 4-mfds

If X_1 & X_2 are simply connected,

Freedman's thm shows that

$$Q_{X_1} \underset{/\mathbb{Z}}{\cong} Q_{X_2} \Rightarrow X_1 \underset{\text{homeo}}{\simeq} X_2 .$$

(一般に, X が smooth なら Kirby-Siebenmann $= 0$ なので, 上で X_1 と X_2 があらかじめ smooth と仮定されていれば交叉形式のみで homeo なことが必要.)

But if SW inv. of X_1 & X_2 are different.

we can conclude $X_1 \underset{\text{diff.}}{\not\cong} X_2$

eg.

$$K3 \# (-\mathbb{CP}^2) \underset{\text{homeo}}{\approx} \overset{3}{\#} \mathbb{CP}^2 \# \overset{20}{\#} (-\mathbb{CP}^2).$$

But,

$$\begin{cases} SW(\underbrace{K3 \# (-\mathbb{CP}^2)}_{\text{a Kähler} \Rightarrow \text{symp.}}, \mathfrak{s}_{can}) = 1 & \text{(Taubes)} \\ SW(\# \mathbb{CP}^2 \# (-\mathbb{CP}^2), \mathfrak{s}) = 0 & (\forall \mathfrak{s}). \end{cases}$$

connected sum.

$\therefore$ They are not diffeo. $\lrcorner$

- <u>Obstruction to a positive scalar curvature metric</u>

X : a smooth mfd

g : a Riem. metric on X

$\rightsquigarrow$ $s_g : X \to \mathbb{R}$: the scalar curvature.

<u>Thm.</u> (Kazdan–Warner '75)

Assume X is closed and $\dim X \geq 3$.

Then,

$\forall f : X \to \mathbb{R}$: smooth function,

$\exists g$: metric on X s.t. $s_g = f$.

$\Longleftrightarrow \exists g$ s.t. $s_g > 0$ everywhere

(called a PSC metric)

Thm. (Witten '94)

X : ori. closed smooth 4-mfd,
w/ $b^+(X) \geq 2$.

If $\exists\, \mathfrak{s}$ on X s.t. $SW(X, \mathfrak{s}) \neq 0$,
then X doesn't admit a PSC metric.

e.g. $\forall$ closed symp. 4-mfd w/ $b^+ \geq 2$
doesn't admit a PSC metric. $\lrcorner$

Other applications :

(1) Genus bound problem

& Thom conjecture (Kronheimer–Mrowka)

$\Sigma^2 \hookrightarrow \mathbb{C}P^2$ w/ $[\Sigma] = d[\mathbb{C}P^1]$ $(d \in \mathbb{Z} \setminus \{0\})$

$$\Rightarrow g(\Sigma) \geq \frac{(d-1)(d-2)}{2}$$

"=" is attained by algebraic curves.

(2) Constraint to intersection forms.

- Alternative proof of Donaldson's diagonalization

  ($\underline{\text{e.g.}}$ $\overset{n}{\oplus} E_8$ $(n>0)$ cannot realized as the intersection form of smooth X.)

- 10/8 - inequality (Furuta) X : spin

$$\Rightarrow b_2(X) \geq \frac{5}{4}|\mathrm{sign}(X)| + 2.$$

• Genus bound problem

Q.

X : ori. closed 4-mfd

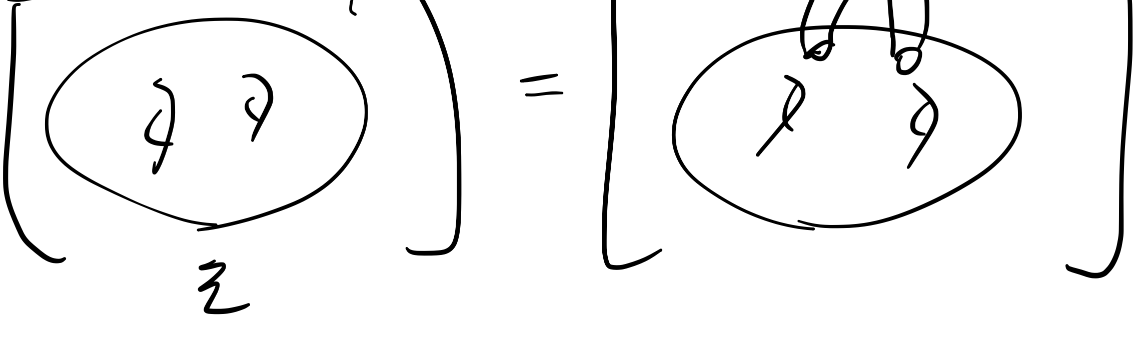
$\alpha \in H_2(X;\mathbb{Z})$.

$\Sigma \hookrightarrow X$: ori. closed, connected embedded surface

$$[\Sigma] = \alpha \Rightarrow g(\Sigma) \geq \boxed{?}.$$

in terms of α.

Rem.



Thm. (Kronheimer–Mrowka '94)

X : ori. closed 4-mfd w/ $b^+(X) \geq 2$

$\mathfrak{s}$ on X s.t. $SW(X, \mathfrak{s}) \neq 0$

$\Rightarrow \max\{-\chi(\Sigma), 0\}$

$\geq |c_1(\mathfrak{s}) \cdot [\Sigma]| + [\Sigma]^2$

$\forall \Sigma \hookrightarrow X$ w/ $[\Sigma]^2 \geq 0$.

$$\begin{array}{ccc} Spin^c(4) & \longrightarrow & U(1) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [q_+, q_-, \lambda] & \longmapsto & \lambda^2 \end{array}$$

$$\begin{array}{ccc} P & & L \\ \downarrow Spin^c(4) & \rightsquigarrow & \downarrow \mathbb{C} \\ X & & X \end{array}$$

$c_1(\mathfrak{s}) := c_1(L)$.

e.g. (obtained from "$b^+=1$" analogue)

$X = \mathbb{C}P^2$

$\Sigma \hookrightarrow \mathbb{C}P^2$ w/ $[\Sigma] = d[\mathbb{C}P^1]$ $(d>0)$.

$\Rightarrow g(\Sigma) \geq \frac{(d-1)(d-2)}{2}$ (Thom conjecture)

"=" is attained by an algebraic curve. $\lrcorner$

- 10/8 - ineq.

if X is simply connected.

$Q_X(\alpha, \alpha) \equiv 0 \mod 2$

$\forall \alpha \in H^2(X;\mathbb{Z})$

Thm. (Furuta '01)

X : ori. closed spin 4-mfd

w/ $b_2(X) \neq 0$

$$\Rightarrow \quad b_2(X) \geq \frac{5}{4} |\underbrace{\mathrm{sign}(X)}_{\substack{\| \\ b^+(X) - b^-(X)}}| + 2.$$

"=" is attained by K3.

Conj (Matsumoto)

$$\Rightarrow \quad b_2(X) \geq \frac{11}{8} |\mathrm{sign}(X)|.$$

§3. SW equations

X : an oriented closed smooth 4-mfd

w/ a Riemannian metric

$\mathfrak{s} = (P, \psi)$: a spin^c str. on X.

$$\left(\begin{array}{c} P \\ \downarrow \mathrm{Spin}^c(4) \\ X \end{array} , \quad P \times_{\mathrm{Spin}^c(4)} SO(4) \xrightarrow[\cong]{\psi} Fr_X . \right)$$

$$\mathrm{Spin}^c(4) = \frac{Sp(1) \times Sp(1) \times U(1)}{\pm 1} \begin{array}{l} \xrightarrow{s_+} U(2) \\ \xrightarrow{s_-} U(2) \\ \xrightarrow{\det} U(1) \end{array}$$

$$\left\{ \begin{array}{l} s_\pm([q_+, q_-, \lambda]) := \underbrace{q_\pm}_{\in} \cdot \lambda , \\ \qquad\qquad Sp(1) \cong SU(2) \qquad (\alpha, \beta \in \mathbb{C}, \; |\alpha|^2 + |\beta|^2 = 1) \\ \qquad\qquad \alpha + j\beta \leftrightarrow \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \\ \det([q_+, q_-, \lambda]) := \lambda^2 \end{array} \right.$$

$$\rightsquigarrow \begin{array}{ccccc} S^+ & & S^- & & \det \\ \downarrow \mathbb{C}^2 & & \downarrow \mathbb{C}^2 & & \downarrow \mathbb{C} \\ X & , & X & , & X \end{array} \text{ w/ hermitian metrics.}$$

$$\mathcal{A}(\mathfrak{s}) = \mathcal{A} := \left\{ U(1)\text{-connections of } \begin{array}{c} \det \\ \downarrow \mathbb{C} \\ X \end{array} \right\}$$

$$\mathcal{C}(\mathfrak{s}) = \mathcal{C} := \mathcal{A} \times \Gamma(S^+).$$

- <u>Dirac operator</u>

Fix $A \in \mathcal{A}$.

$\rightsquigarrow \not{D}_A : \Gamma(S^+) \to \Gamma(S^-)$ is defined as

$$\not{D}_A : \Gamma(S^+) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S^+) \xrightarrow[\text{multi}]{\text{Clifford}} \Gamma(S^-).$$

A : $U(1)$-conn. on $\det$.

$\longleftrightarrow \{A_\alpha\}$, $A_\alpha \in \Omega^1(U_\alpha, i\mathbb{R})$ w/ $A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + A_\alpha$

$X = \bigcup_\alpha U_\alpha$ $\quad\left(g_{\alpha\beta} : U_{\alpha\beta} \to U(1) \text{ : transition functions of } \det \right)$

$$\mathfrak{spin}^c(4) \cong \mathfrak{sp}(1) \times \mathfrak{sp}(1) \times \mathfrak{u}(1) \cong \mathfrak{so}(4) \times \mathfrak{u}(1).$$

$$\uparrow$$

$$Spin^c(4) = \frac{Sp(1) \times Sp(1) \times U(1)}{\pm 1}$$

$$\frac{Sp(1) \times Sp(1)}{\pm 1} = SO(4)$$

$$\begin{cases} \text{Levi-Civita conn } (\rightsquigarrow \text{ locally } \mathfrak{so}(4)\text{-valued 1-forms}) \\ \{A_\alpha\}_\alpha : \mathfrak{u}(1)\text{-valued 1-forms} \end{cases}$$

$\Rightarrow$ $\mathfrak{spin}^c(4)$-valued 1-forms

These give a conn. on P

$\rightsquigarrow$ $\nabla_A : \Gamma(S^+) \to \Gamma(T^*X \otimes S^+)$

: the covariant derivative of the connection on S^+ induced from P.

- <u>Clifford multiplication</u>

It's a map $TX \times S^+ \xrightarrow{c} S^-$

modeled by
$$\begin{array}{ccc} \mathbb{H} \times \mathbb{H} & \longrightarrow & \mathbb{H}. \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (q, q') & \longmapsto & q \cdot q' \end{array}$$

i.e.
$$\begin{array}{ccc} TX \times S^+ = \left(\dfrac{P \times \mathbb{R}^4}{SO(4)}\right) \times \left(\dfrac{P \times \mathbb{R}^4}{U(2)}\right) & \longrightarrow & \dfrac{P \times \mathbb{R}^4}{U(2)} = S^- \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \left([u, q], [u, q']\right) & \longmapsto & [u, qq'] \end{array}$$

(Regard $\mathbb{R}^4 = \mathbb{H}$.)

Similary, $TX \otimes S^- \xrightarrow{c} S^+$ is defined

by
$$\begin{array}{ccc} \mathbb{H} \times \mathbb{H} & \longrightarrow & \mathbb{H}. \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (q, q') & \longmapsto & -\bar{q} q' \end{array}$$

Remark.

$$a \in TX \rightsquigarrow \rho(a) \in \mathrm{End}_{\mathbb{C}}(S^+ \oplus S^-)$$

$$\begin{aligned} \rho(a) : S^+ \oplus S^- &\longrightarrow S^+ \oplus S^- \\ \Phi &\longmapsto c(a, \Phi) \end{aligned}$$

$$a \wedge b \in \Lambda^2 TX \rightsquigarrow \rho(a \wedge b)$$
$$:= \frac{1}{2}\left(\rho(a)\rho(b) - \rho(b)\rho(a)\right)$$
$$\in \mathrm{End}_{\mathbb{C}}(S^\pm).$$

- Self-duality

$$* : \Lambda^i \to \Lambda^{4-i} : \text{Hodge } *\text{-op.}$$
$$\Lambda^i = \Lambda^i T^*X$$

$$\omega = \omega^+ + \omega^-$$

$$\underset{=1}{**} : \Lambda^2 \circlearrowleft \rightsquigarrow \Lambda^2 = \underbrace{\Lambda^+}_{\substack{+1 \\ \text{eigensp}}} \oplus \underbrace{\Lambda^-}_{\substack{-1 \\ \text{eigensp.}}}$$

SW equations

For $(A, \Phi) \in \mathcal{C}$,

$$\begin{cases} \rho(F_A^+) = \sigma(\Phi), \\ \not{D}_A \Phi = 0 \end{cases},$$

where,

$$\begin{cases} F_A \in \underbrace{i\mathbb{R}}_{} \otimes \Omega^2 = i\Omega^2 : \text{the curvature of } A. \quad (i\mathbb{R} = \mathfrak{u}(1)) \\ \sigma(\Phi) = \Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} id_{S^+} \in End_0(S^+), \end{cases}$$

This is a non-linear equasion:

$$\begin{cases} \sigma(\Phi) : \text{non-linear} \\ \not{D}_{A+a} \Phi = D_A \Phi + \underbrace{a \cdot \Phi}_{\text{non-linear.}} \end{cases}$$

(But F_A^+ is linear!)

Fact

$$\rho : \Lambda^+ \xrightarrow{\cong} \overset{\sigma(\Phi)\ \in}{su(S^+)} \subset \mathrm{End}_{\mathbb{C}}(S^+)$$

(So we write F_A^+ for $\rho(F_A^+)$.)

The solutions to the SW eq is the zero set of

$$SW : \underset{(A,\Phi)}{\overset{\cup\!\!\!\vert}{\mathcal{C}}} \to \overset{\cup\!\!\!\vert}{\mathcal{D}} := i\Omega^+ \oplus \Gamma(S^-)$$

$$(A,\Phi) \mapsto (F_A^+ - \sigma(\Phi),\ D\!\!\!\!/_A \Phi)$$

- Action of the gauge group

$$\mathcal{G} := \mathrm{Map}(X, U(1)).$$

$$\underset{u}{\mathcal{G}} \curvearrowright \underset{(A,\Phi)}{\mathcal{C}} \quad ; \quad u\cdot(A,\Phi) := (A - 2u^{-1}du,\ u\cdot\Phi)$$

$$\mathcal{G} \curvearrowright \underset{(\mu,\Psi)}{\mathcal{D}} \quad ; \quad u\cdot(\mu,\Psi) := (\mu,\ u\cdot\Psi)$$

Lem.

$SW : \mathcal{C} \to \mathcal{D}$ is $\mathcal{G}$-equivariant.

$\rightsquigarrow \mathcal{G} \curvearrowright SW^{-1}(0).$

Def.

$\mathcal{M} := SW^{-1}(0)/\mathcal{G}$ is called the <u>moduli space</u>.

(Recall : The SW inv. is defined
as "$\#\mathcal{M}$" $\in \mathbb{Z}$.)

§4. Properties of $\mathcal{M}$.

$\mathcal{B} := \mathcal{C}/\mathcal{G}$.

Is $\mathcal{G} \curvearrowright \mathcal{C}$ free?

$\rightsquigarrow$ No!

Lem.

$(A, \Phi) \in \mathcal{C}$.

$\mathrm{Stab}_{(A,\Phi)}(\mathcal{G}) = \{1\}$

$\Longleftrightarrow \Phi \not\equiv 0$.

Def.

$(A, \Phi) \in \mathcal{C}$ is called <u>irreducible</u>

if $\Phi \not\equiv 0$.

$\mathcal{C}^* := \mathcal{C} \setminus \mathcal{A} \times \{0\}$, $\mathcal{B}^* := \mathcal{C}^*/\mathcal{G}$.

Lem.

(After suitable Sobolev completion of $\mathcal{C}$ & $\mathcal{G}$) $\mathcal{B}^*$ is Hausdorff & a Hilbert manifold.

$\uparrow$ ∞-dim. analogy of :

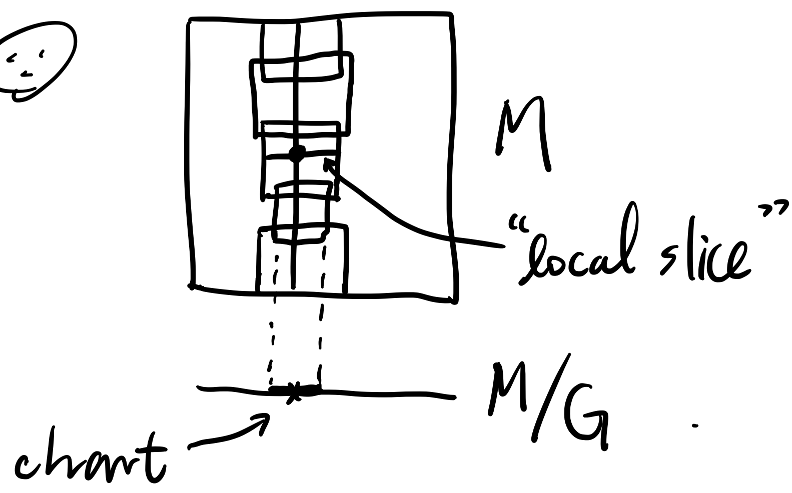
Recall (local slice theorem)

G : a cpt Lie group

M : a (fin. dim.) mfd

$G \curvearrowright M$: freely.

$\Rightarrow M/G$ is Hausdorff & a Hilbert mfd.



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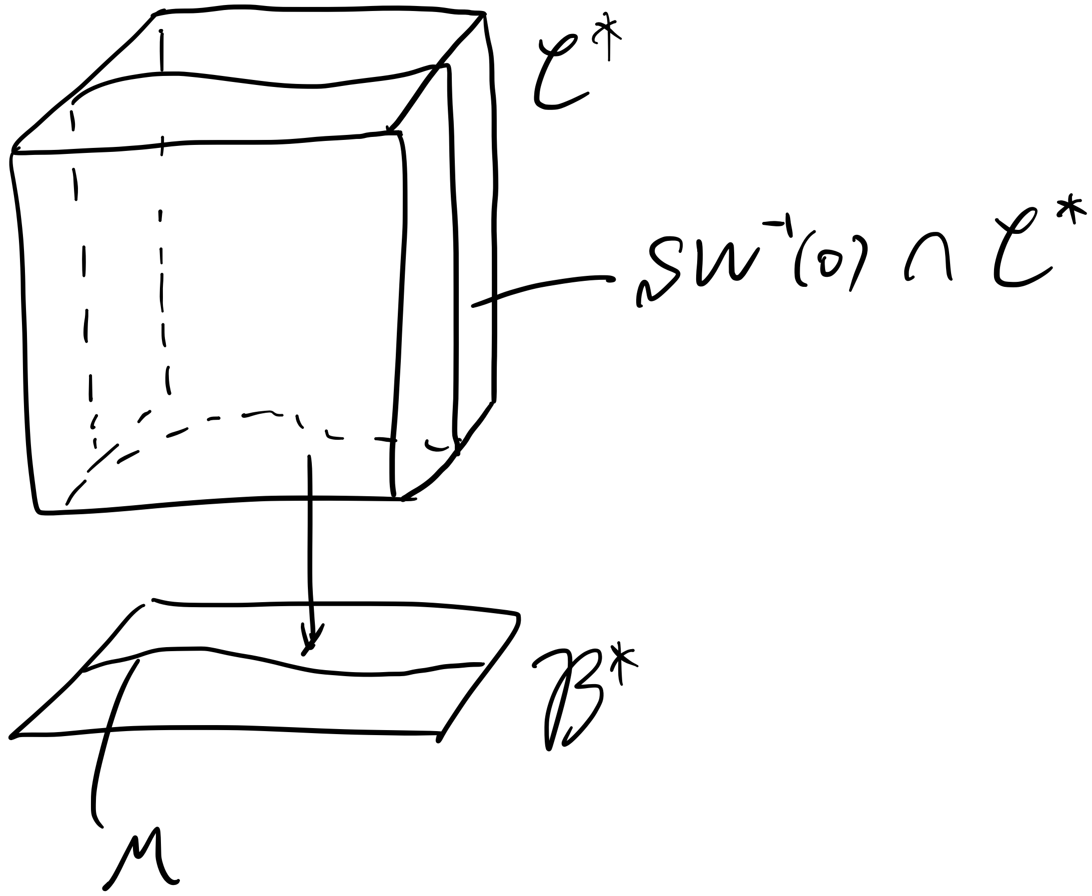
Need :

$U(1)$ is cpt !



$$\underline{\text{Assume}}\quad d(SW)_{(A,\Phi)} : \underset{\cong\, i\Omega^1 \oplus \Gamma(S^+)}{T_{(A,\Phi)}\mathcal{C}} \to T_{(0,0)}\mathcal{D} \cong \mathcal{D}$$

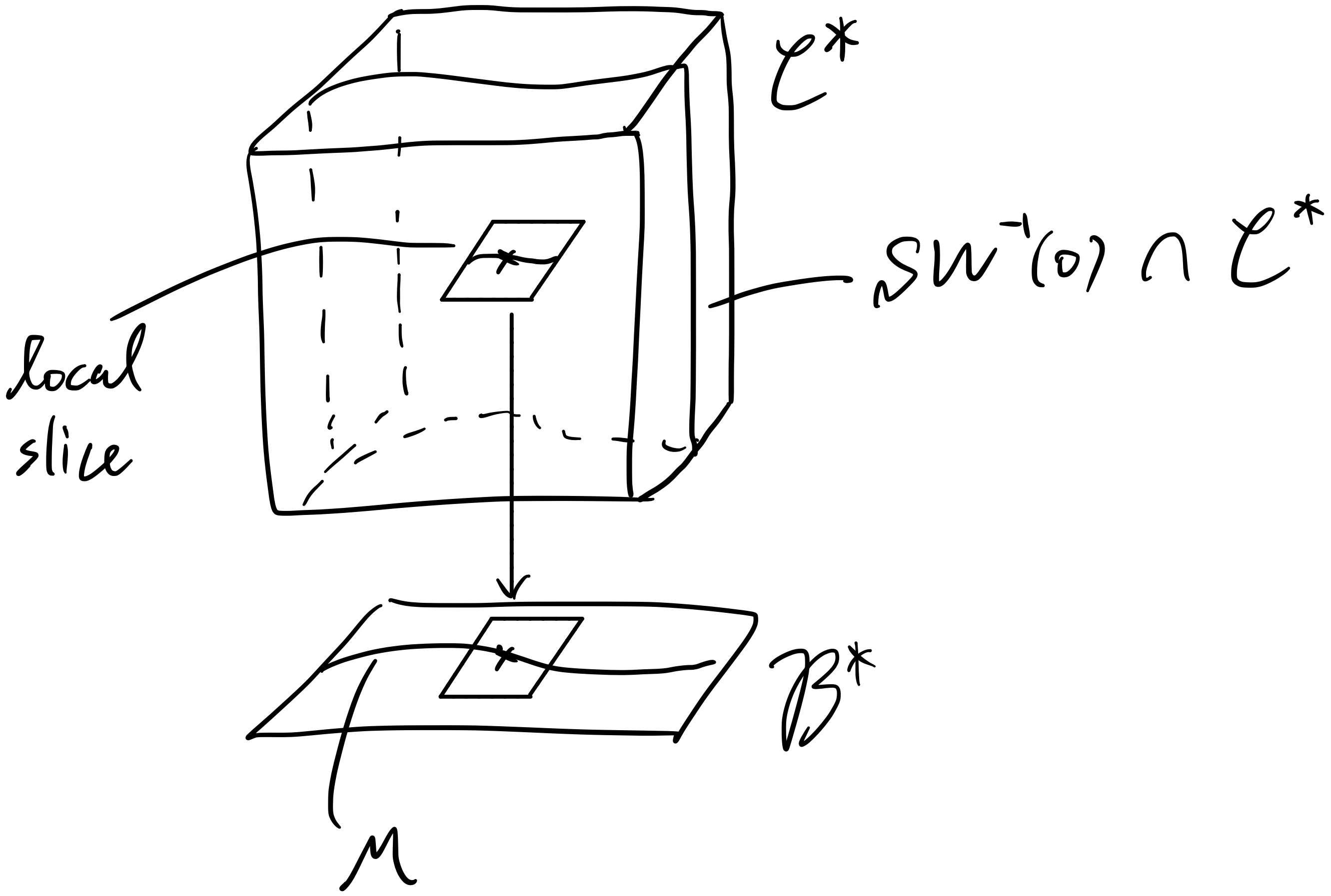
is surjective for $\forall (A,\Phi) \in SW^{-1}(0)$.

Then $SW^{-1}(0)$ is a mfd (∞-dim).

We will get rid of this assumption by perturbing the SW eq.

To do this implicit function thm, we need Sobolev completion.

By restricting the local slice, we get a local slice for $SW^{-1}(0)$:



$\Rightarrow$ $\mathcal{M}$ is also a mfd !

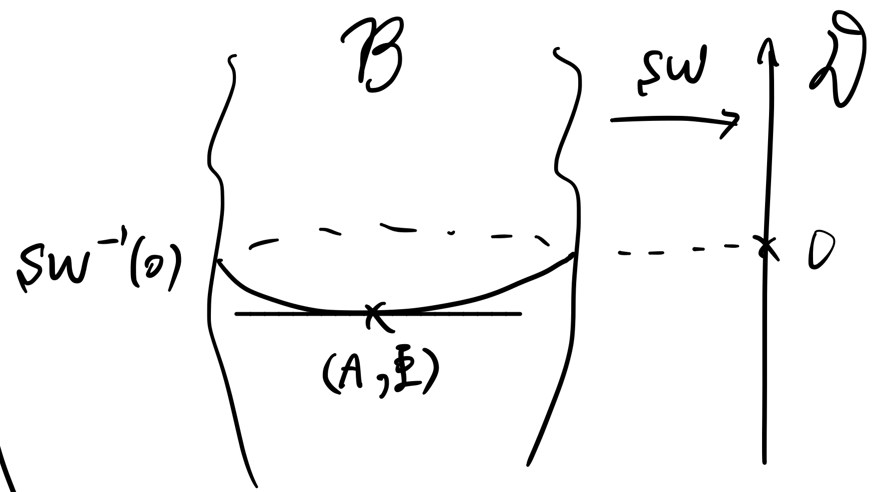
In fact $\mathcal{M}$ is a finite dim. mfd.

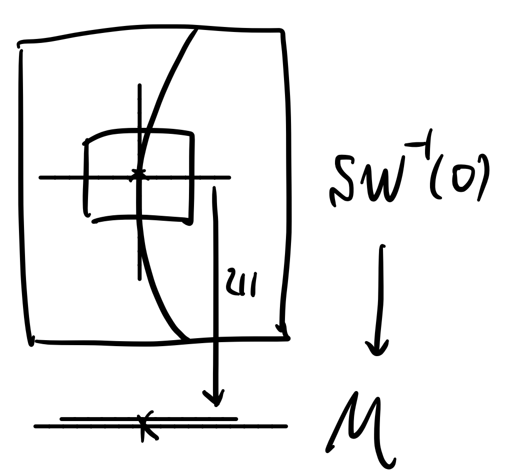
- <u>Linearization of the SW eq.</u>

$$T_{(A,\Phi)}(SW^{-1}(0)) \cong \operatorname{Ker}(dSW_{(A,\Phi)}),$$

$$T_{[A,\Phi]}\mathcal{M} \cong \operatorname{Ker}(dSW_{(A,\Phi)}) / \operatorname{Im}(dG(A,\Phi))$$

$$G(A,\Phi) : \mathcal{G} \to \mathcal{C}$$
$$u \mapsto u\cdot(A,\Phi)$$





$$
\begin{aligned}
SW : \mathcal{C} &\longrightarrow \mathcal{D} \\
(A, \Phi) &\longmapsto (F_A^+ - \sigma(\Phi), \not{D}_A \Phi)
\end{aligned}
$$

$$
\begin{aligned}
\rightsquigarrow dSW_{(A,\Phi)} : T_{(A,\Phi)}\mathcal{C} = i\Omega^1 \oplus \Gamma(S^+) &\longrightarrow i\Omega^+ \oplus \Gamma(S^-) \cong T_{(0,0)}\mathcal{D} \\
(a, \phi) &\longmapsto \Big(d^+ a + \binom{\text{0-th}}{\text{order}}, \\
&\qquad \not{D}_A \phi + \binom{\text{0-th}}{\text{order}}\Big)
\end{aligned}
$$

For a fixed $(A, \Phi) \in \mathcal{C}$,

$$
\begin{aligned}
G(A,\Phi) : \mathcal{G} &\longrightarrow \mathcal{C} \\
u &\longmapsto (A - 2u^{-1}du, u \cdot \Phi)
\end{aligned}
$$

$$
\begin{aligned}
\rightsquigarrow dG(A,\Phi)_1 : \underset{T_1\mathcal{G}}{\underset{\shortparallel}{Map(X, U(1))}} = i\Omega^0 &\longrightarrow i\Omega^1 \oplus \Gamma(S^+) . \\
f &\longmapsto \Big(-2df, \binom{\text{0-th}}{\text{order}}\Big)
\end{aligned}
$$

$\therefore$ $T_{[A,\Phi]}\mathcal{M}$ is the middle cohomology of

$$i\Omega^0 \xrightarrow{\delta G(A,\Phi)} i\Omega^1 \oplus \Gamma(S^+) \xrightarrow{d\,SW} i\Omega^+ \oplus \Gamma(S^-) \quad \cdots \circledast$$

$$f \longmapsto (-2df,\ 0\text{th})$$

$$(a,\phi) \longmapsto \left(d^+a + 0\text{th},\ \cancel{D}_A\phi + 0\text{th}\right).$$

If $(A,\Phi) \in SW^{-1}(0)$, $\circledast$ is a complex.

[$\because$ $SW^{-1}(0)$ is $\mathcal{G}$-inv.]

$\circledast$ is the elliptic complex!

($\because$ The leading terms are the direct sum of

$$\& \begin{cases} \Omega^0 \xrightarrow{-2d\cdot} \Omega^1 \xrightarrow{d^+\cdot} \Omega^+, \\ \Gamma(S^+) \xrightarrow{\cancel{D}_A\cdot} \Gamma(S^-) \end{cases}$$

)

The dim. of the middle cohomology is therefore calculated from the index thm:

$$d(\mathfrak{s}) := \frac{1}{4}\left(c_1(\mathfrak{s})^2 - 2\chi(X) - 3\underbrace{\mathrm{sign}(X)}_{b^+(X) - b^-(X)}\right)$$

is the dimension.

(called the formal dim.)

[Comment:

$$\mathrm{ind}\left(\Omega^1 \xrightarrow{d^* + d^+} \Omega^0 \oplus \Omega^+\right)$$
$$= -(b_0 - b_1 + b^+)$$

is computed only by Hodge theory.

$$\mathrm{ind}_{\mathbb{C}}\left(\Gamma(S^+) \xrightarrow{\not{D}_A} \Gamma(S^-)\right)$$
$$= \int_X \hat{A}(X) \wedge e^{\frac{1}{2}c_1(\mathfrak{s})}$$
$$= \int_X \left(1 - \frac{1}{24}p_1(X)\right)\left(1 + \frac{1}{2}c_1(\mathfrak{s}) + \frac{1}{8}c_1(\mathfrak{s})^2\right)$$

$$= -\int_X \frac{p_1(X)}{24} + \frac{1}{8} c_1(\mathfrak{s})^2$$

$$= -\frac{1}{3}\operatorname{sign}(X) + \frac{1}{8} c_1(\mathfrak{s})^2.$$

Summarized :

Prop.

If $dSW_{(A,\Phi)}$ is surj $\left((A,\Phi) \in SW^{-1}(0) \right)$,

then $\dim T_{[A,\Phi]} \mathcal{M} = d(\mathfrak{s})$.

Thm. (Witten)

$\mathcal{M}$ is cpt.

Weitzenböck formula

Prop.

An orientation of $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$ determines an orientation of $\mathcal{M}$.

Want.

A nowhere vanishing section of

$$\begin{array}{c} \mathbb{R} \to \det T\mathcal{M} \\ \downarrow \\ \mathcal{M} \end{array}$$

parametrized by $\mathcal{B}^*$.

$\det($linearization of the SW eq & gauge$)$

$= \det(\mathrm{Ker}(\quad)) \otimes \det(\mathrm{Coker}(\quad))$.

§5. Definition of the SW invariant.

X : an ori. closed 4-mfd
w/ a Riem. metric g.

$\mathfrak{s}$: a spin^c str. on X.

Fix an ori. of $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$.

$$\rightsquigarrow \quad SW : \begin{tikzcd} \mathcal{C} \arrow[r] \arrow[loop above, "\mathcal{G}"] & \mathcal{D} \arrow[loop above, "\mathcal{G}"] \\ \mathcal{C}^* \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

$$\mathcal{M} := SW^{-1}(0)/\mathcal{G}$$
$$\cap$$
$$\mathcal{B} := \mathcal{C}/\mathcal{G} \supset \mathcal{C}^*/\mathcal{G} =: \mathcal{B}^*.$$

Assume:

$$(\star)\begin{cases} dSW_{(A,\Phi)} \text{ is surj } (\forall (A,\Phi) \in SW^{-1}(0)), \\ \mathcal{M} \subset \mathcal{B}^*. \end{cases}$$

Then M is an oriented cpt mfd of dim $d(\mathfrak{s}) = \frac{1}{4}\left(c_1(\mathfrak{s})^2 - 2\chi(X) - 3\,\mathrm{sign}(X)\right)$.

<u>Case of $d(\mathfrak{s}) < 0$:</u>

Then $M = \phi$.

Define $SW(X, \mathfrak{s}) := 0 \in \mathbb{Z}$.

<u>Case of $d(\mathfrak{s}) = 0$:</u>

Define $SW(X, \mathfrak{s}) := \# M$

$$= \langle 1, [M] \rangle.$$

For the case when $d(\mathfrak{s}) > 0$,
we use some cohomology class on $\mathcal{B}^*$.

Fix $x_0 \in X$.

$$\mathcal{G}_0 := \{ u \in \mathcal{G} \mid u(x_0) = 1 \}$$

$$1 \longrightarrow \mathcal{G}_0 \hookrightarrow \mathcal{G} \longrightarrow S^1 \longrightarrow 1.$$

$$\begin{tikzcd}
S^1 \arrow[r] & \mathcal{C}^*/\mathcal{G}_0 \arrow[d] \arrow[r, leftrightarrow] & \mathbb{C} \arrow[r] & \mathcal{L} \arrow[d] \\
 & \mathcal{B}^* & & \mathcal{B}^*
\end{tikzcd}$$

<u>Case of $d(\mathfrak{s}) < 0$</u> :

If $d(\mathfrak{s}) = -2d \quad (\exists d > 0)$,

$$SW(X, \mathfrak{s}) := \langle c_1(\mathcal{L})^d, [M] \rangle \in \mathbb{Z}.$$

If $d(\mathfrak{s})$ is odd, $SW(X, \mathfrak{s}) := 0$.

Thm.

If $b^+(X) \geq 1$, we can "perturb" the SW eq to satisfy Assumption ⊛, and if $b^+(X) \geq 2$, $SW(X, \mathfrak{s}) \in \mathbb{Z}$ is indep. of choices of metric & perturbations.

(Use the "perturbed moduli space" instead of $\mathcal{M}$.)

∘ Perturbed SW eq.

$Met(X) := \{$ Riem. metrics on $X \}$

$$\begin{array}{l} \Pi(X) := \coprod_{g \in Met(X)} \Omega^+_g(X) \\ \pi \downarrow \\ Met(X) \end{array}$$

: the space of perturbations.

Take $\mu \in \Pi(X)$, set $g := \pi(\mu)$.

$(A, \Phi) \in \mathcal{C} = \mathcal{C}_g.$

$$\begin{cases} F_A^+ = \sigma(\Phi) + i\mu, \\ \not{D}_A \Phi = 0 \end{cases}$$

the perturbed SW eq.

<u>Def.</u>

(A, Φ, μ) is irr

$\Leftrightarrow \Phi \neq 0$,

<u>Q</u>. when $\exists$ red. sol?

(A, Φ, μ) : red sol.

$\Leftrightarrow F_A^+ = i\mu.$

Fix $A_0 \in \mathcal{A}$.

Then, $A = A_0 + a \quad (\exists a \in i\Omega^1)$.

$$F_A^+ = i\mu \iff d^+a + F_{A_0}^+ = i\mu.$$

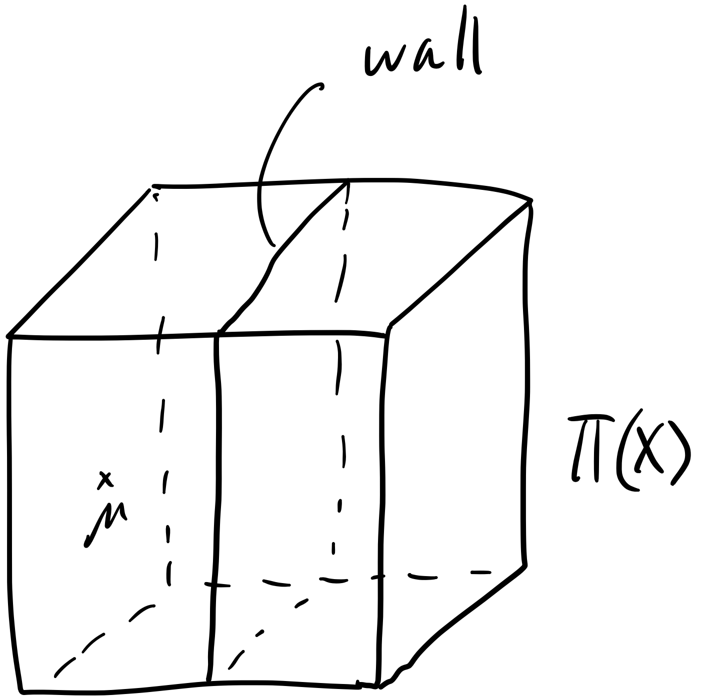
$\therefore \; i\mu \in \underbrace{\mathrm{Im}\, d^+ + F_{A_0}^+}$ ← called <u>the wall</u>

$\iff$ red. sol.

<u>Lem.</u>

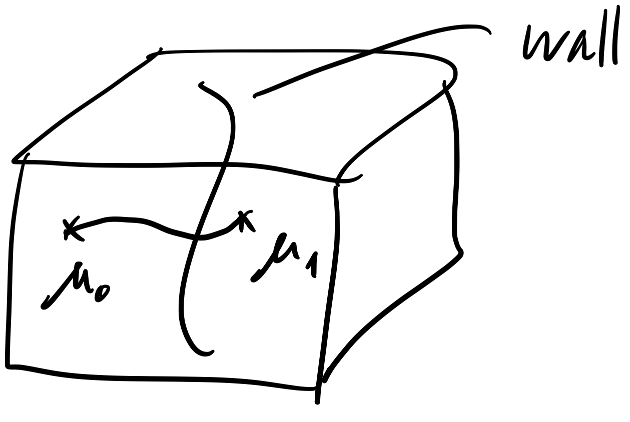
$\Omega^+ = \mathrm{Im}\, d^+ \oplus \underbrace{\mathcal{H}^+}_{\text{s.D. harmonic 2-forms.}}, \quad \dim \mathcal{H}^+ = b^+.$

$\therefore \; \mathrm{Im}\, d^+$ is of codim b^+ !



So, if $b^+ \geq 1$, we can take $\mu \in \Pi(X)$ avoiding the wall.

(If $b^+ \geq 2$, two $\mu_0, \mu_1 \in \Pi(X) \setminus \text{wall}$ can be connected by a path in $\Pi(X) \setminus \text{wall}$.)



For $\mu \in \Pi(X)$, set $g = \pi(\mu)$ and

$$
\begin{array}{rccc}
SW_\mu : & \mathcal{C}_g & \longrightarrow & \mathcal{D}_g \\
& \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
& (A, \Phi) & \longmapsto & (F_A^+ - \sigma(\Phi) - i\mu,\ D\!\!\!\!/_A \Phi)
\end{array}
$$

$\mathcal{G} \curvearrowright \Pi(X)$: trivially

$$\mathcal{M}_\mu := SW_\mu^{-1}(0)/\mathcal{G}.$$

If $b^+ \geq 1$, by taking $\mu \in \Pi(X) \setminus$ wall,

we get $\mathcal{M}_\mu \subset \mathcal{B}^*$.

(Half of Assumption ⊛)

Lem.

Fix $g \in Met(X)$.

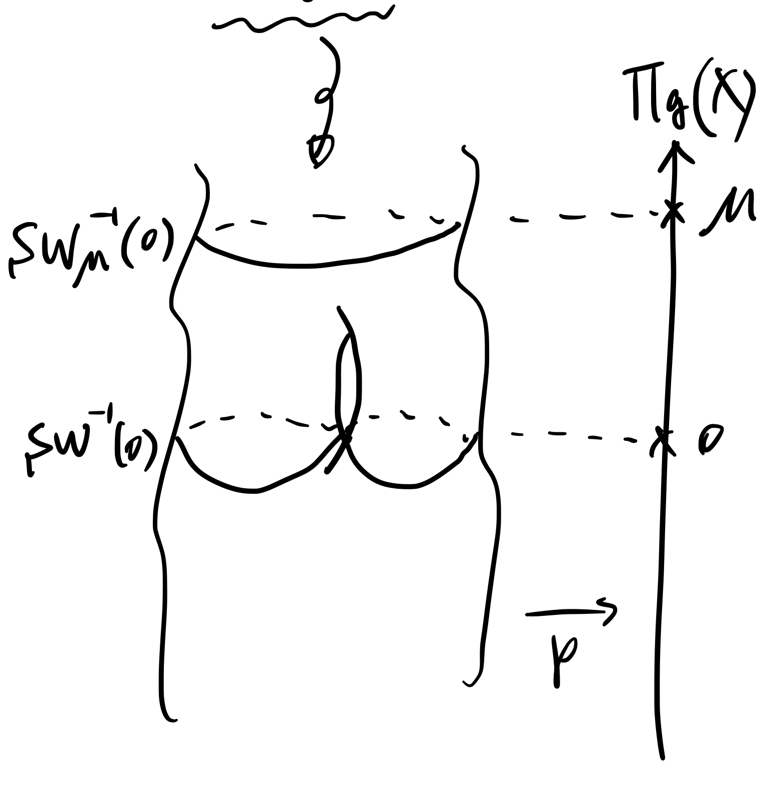
Consider

$$\widetilde{SW}_g : \mathcal{C}_g \times \overset{\Omega^+_g(X)}{\overset{\parallel}{\Pi_g(X)}} \longrightarrow \mathcal{D}_g .$$

$$(A, \Phi, \mu) \longmapsto (F_A^+ - \sigma(\Phi) - i\mu \, , \, D_A \Phi)$$

Then $(d\widetilde{SW}_g)_{(A,\Phi,\mu)}$ is surj at any $(A, \Phi, \mu) \in \widetilde{SW}_g^{-1}(0)$.

So $\widetilde{SW}_g^{-1}(0)$ is a (∞-dim) mfd.



By Sard's thm, for a generic $\mu \in \Pi_g(X)$, $SW_\mu^{-1}(0)$ is a mfd.

Combine the consideration about reducibles!

$\rightsquigarrow$ If $b^+ \geq 1$, for a "generic" $\mu \in \Pi(X) \setminus \text{wall}$,

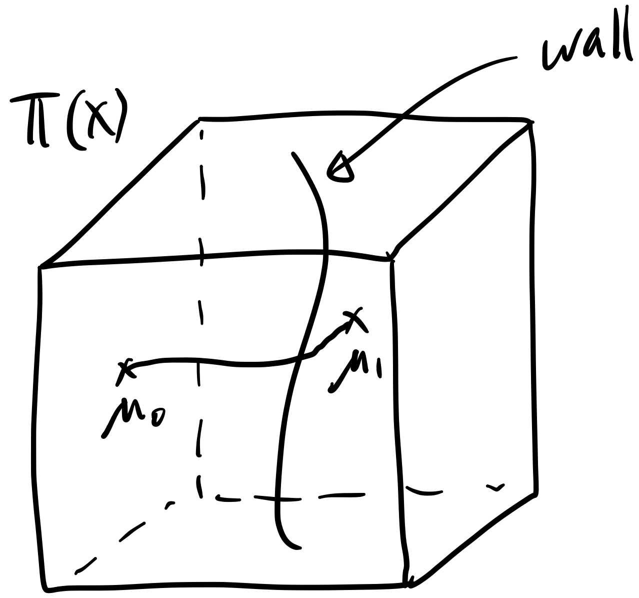
Assumption (A) is satisfied for the perturbed SW eq w.r.t. μ.

Invariance

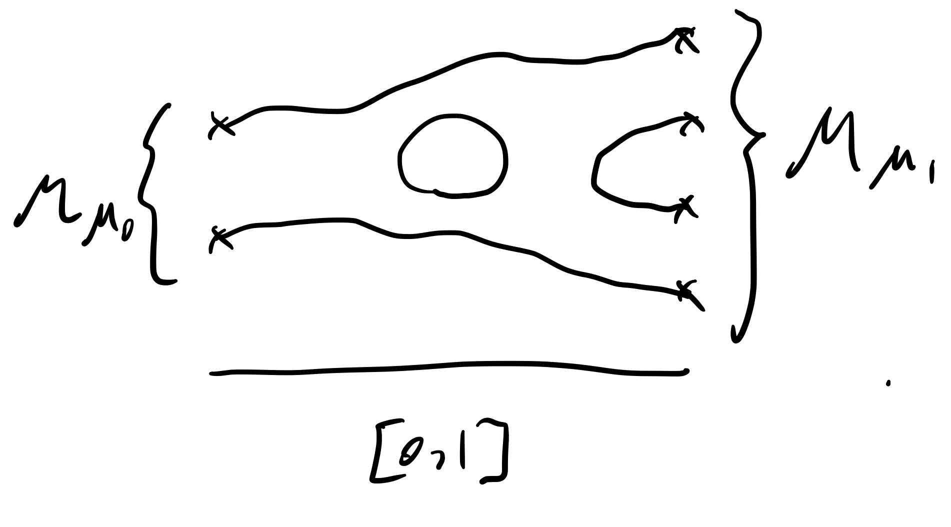
Assume $b^+ \geq 2$.

Take generic $\mu_0, \mu_1 \in \Pi(X) \setminus \text{wall}$.

Take a generic path γ from μ_0 to μ_1 in $\Pi(X) \setminus \text{wall}$.



γ induces a cobordism from M_{μ_0} to M_{μ_1}:



Stokes's thm

$\Rightarrow \# \mathcal{M}_{\mu_0} = \# \mathcal{M}_{\mu_1}$