The Hitchin fibration and real forms through spectral data

Laura Schaposnik
Ruprecht-Karls Universität Heidelberg

School of Mathematics
University of Adelaide
June 2013
# The Plan

1. **Higgs bundles.**
   - Classical Higgs bundles
   - $G^c$-Higgs bundles.

2. **The Hitchin fibration**
   - Spectral data approach.
   - G-Higgs bundles as fixed points.

3. **Spectral data approach for real forms**
   - Discrete fixed point set.
   - Pos. dim. fixed point set.
   - No fixed point set.

4. **Applications**
   - Connected components.
   - Topological invariants.

Based on:
- arXiv:1111.2550
- Spectral data also used in arXiv:1305.4638 with D. Baraglia.
CLASSICAL HIGGS BUNDLES

Σ compact Riemann surface of genus $g > 2$, canonical bundle $K = T^*\Sigma$.

Definition

A Higgs bundle on a compact Riemann surface $\Sigma$ of genus $g > 1$, is a pair $(E, \Phi)$ for $E$ a holomorphic vector bundle on $\Sigma$ and $\Phi$ a section in $H^0(\Sigma, \text{End}(E) \otimes K)$. See example.
**CLASSICAL HIGGS BUNDLES**

\( \Sigma \) compact Riemann surface of genus \( g > 2 \), canonical bundle \( K = T^* \Sigma \).

**Definition**

A *Higgs bundle* on a compact Riemann surface \( \Sigma \) of genus \( g > 1 \), is a pair \((E, \Phi)\) for \( E \) a holomorphic vector bundle on \( \Sigma \) and \( \Phi \) a section in \( H^0(\Sigma, \text{End}(E) \otimes K) \). See example.

The *slope* of a vector bundle \( F \) be \( \mu := \deg(F)/\text{rk}(F) \). A Higgs bundle \((E, \Phi)\) is

- **stable** if for each \( \Phi \)-invariant subbundle \( F \) one has \( \mu(F) < \mu(E) \);
- **semi-stable** if for each \( \Phi \)-invariant subbundle \( F \) one has \( \mu(F) \leq \mu(E) \);
- **polystable** if \((E, \Phi) = (E_1, \Phi_1) \oplus (E_2, \Phi_2) \oplus \ldots (E_r, \Phi_r)\), where \((E_i, \Phi_i)\) is stable with \( \mu(E_i) = \mu(E) \) for all \( i \).

See example.
$G^c$-Higgs bundles

$G^c$ be a complex semisimple Lie group.

**Definition**

A $G^c$-Higgs bundle is a pair $(P, \Phi)$ where

- $P$ is a principal $G^c$-bundle over $\Sigma$,

- Higgs field $\Phi$ is a holomorphic section of the vector bundle $\text{Ad}P \otimes_\mathbb{C} K$,

for $\text{ad}P$ the vector bundle associated to the adjoint representation.

- Can extend stability notions. Then, call $\mathcal{M}_{G^c}$ the moduli space of $S$-equivalence classes of semi-stable $G^c$-Higgs bundles.
**$G^c$-Higgs bundles: Examples**

- Classical Higgs bundles are given by $GL(n, \mathbb{C})$-Higgs bundles.
$G^c$-Higgs Bundles: Examples

- Classical Higgs bundles are given by $GL(n, \mathbb{C})$-Higgs bundles.
- For $G^c \subset GL(n, \mathbb{C})$ one has classical Higgs bundles + extra conditions:
- Classical Higgs bundles are given by $GL(n, \mathbb{C})$-Higgs bundles.
- For $G^c \subset GL(n, \mathbb{C})$ one has classical Higgs bundles + extra conditions:
  - $SL(n, \mathbb{C})$

\[(E, \Phi) \text{ for } \begin{cases} \text{$E$ rk $n$ vector bundle s.t. } \Lambda^n E \cong \mathcal{O} \\ \Phi : E \rightarrow E \otimes K \text{ s.t. } Tr(\Phi) = 0 \end{cases}\]
**$G^c$-Higgs bundles: Examples**

- Classical Higgs bundles are given by $GL(n, \mathbb{C})$-Higgs bundles.
- For $G^c \subset GL(n, \mathbb{C})$ one has classical Higgs bundles + extra conditions:
  
  - $SL(n, \mathbb{C})$
    
    $$(E, \Phi) \text{ for } \begin{cases} 
    E \text{ rank } n \text{ vector bundle s.t. } \Lambda^n E \cong \mathcal{O} \\
    \Phi : E \to E \otimes K \text{ s.t. } \text{Tr}(\Phi) = 0
    \end{cases}$$
  
  - $Sp(2n, \mathbb{C})$
    
    $$(E, \Phi) \text{ for } \begin{cases} 
    E \text{ rank } 2n \text{ symplectic vector bundle} \\
    \text{symplectic form } \omega \text{ on } E \\
    \Phi : E \to E \otimes K \text{ s.t. } \omega(\Phi v, w) = -\omega(v, \Phi w)
    \end{cases}$$
Let $d_i$, for $i = 1, \ldots, k$, be the degrees of the basic invariant polynomials $p_i$ on the Lie algebra $\mathfrak{g}^c$ of $G^c$.

**Definition**

The Hitchin fibration is

$$h : \mathcal{M}_{G^c} \longrightarrow \mathcal{A}_{G^c} := \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}),$$

$$(E, \Phi) \mapsto (p_1(\Phi), \ldots, p_k(\Phi)).$$

- $h$ is a proper map and $\dim \mathcal{A}_{G^c} = \dim \mathcal{M}_{G^c}/2$.
- The Hitchin map makes the Higgs bundles moduli space into an integrable system.
- For most classical groups (not $SO(2n, \mathbb{C})$), we take polynomials on $\text{Tr}(\Phi^i)$ for a basis of invariant polynomials.

See examples.

N.Hitchin ‘87
Spectral data approach

The idea

\[ \mathcal{M}_{G^c} \rightarrow \mathcal{A}_{G^c} = \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}) \]

\[ (E, \Phi) \rightarrow \text{char}(\Phi) \sim a = (a_1, \ldots, a_k) \]

S a \( d_k \)-fold cover of \( \Sigma \)

Spectral data

Data on \( S \)
SPECTRAL DATA APPROACH: $GL(n, \mathbb{C})$

$h : (E, \Phi) \mapsto \text{char}(\Phi) \in \bigoplus_{i=1}^{n} H^0(\Sigma, K^i) = \mathcal{A}_{GL(n, \mathbb{C})}$

$\text{char}(\Phi) = \eta^n + a_1 \eta^{n-1} + a_2 \eta^{n-2} + \ldots + a_{n-1} \eta + a_n$

So the fibration looks as follows...

N.Hitchin ‘87, ‘07
Spectral data approach: $GL(n, \mathbb{C})$

The construction

Starting with $(S, M)$ we get a stable Higgs bundle $(E, \Phi)$ for

- The rank $n$ vector bundle $E = \rho_* M$;
- The Higgs field $\Phi$ induced by

$$H^0(\rho^{-1}(\mathcal{U}), M) \xrightarrow{\eta} H^0(\rho^{-1}(\mathcal{U}), M \otimes \rho^* K)$$

for an open $\mathcal{U} \subset \Sigma$ by definition of direct image, gives

$$H^0(\mathcal{U}, \rho_* M) \rightarrow H^0(\mathcal{U}, \rho_* M \otimes K)$$

Pushes down to $\Phi : E \rightarrow E \otimes K$. 
Starting with a stable \((E, \Phi)\) we get the spectral data \((S, M)\) for

- The smooth spectral curve \(S\) defined by \(\det(\eta - \rho^*\Phi) = 0\);
- For the line bundle \(U := \text{coker}(\eta - \rho^*\Phi)\), one has \(\rho_*U = E\).

The generic fibre of the Hitchin fibration is isomorphic to the Jacobian of the spectral curve \(S\).
SPECTRAL DATA APPROACH: $SL(n, \mathbb{C})$

As classical Higgs bundles + extra data.

$$\Lambda^n E = O \iff \Lambda^n \rho_* M \cong O$$

From [Beauville-Narasimhan-Ramanan, 1989],

$$\Lambda^n \rho_* M \cong Nm(M) \otimes K^{-n(n-1)/2}.$$  

$$Nm : \text{Pic}(S) \rightarrow \text{Pic}(\Sigma)$$

$$\sum n_i p_i \mapsto \sum n_i \rho(p_i)$$

Then $\Lambda^n \rho_* M = O$ if and only if $Nm(M) \cong K^n(n-1)/2$, or equivalently

$$M \otimes \rho^* K^{-(n-1)/2} \in \text{Ker}(Nm) =: \text{Prym}(S, \Sigma).$$

The generic fibre of the Hitchin fibration is biholomorphically equivalent to the Prym variety Prym$(S, \Sigma)$. 
G-Higgs bundles

Set up

- $G$ a real reductive Lie group;
- $\mathfrak{g}^\mathbb{C}$ complexified Lie algebra of $G$;
- $\mathfrak{m}^\mathbb{C}$ such that
  \[
  \mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C}
  \]
- $H \subset G$ the maximal compact subgroup;
- $\mathfrak{h}^\mathbb{C}$ complexified Lie algebra of $H$;
- $\text{Ad}_{|H^\mathbb{C}} : H^\mathbb{C} \to GL(\mathfrak{m}^\mathbb{C})$ is the isotropy representation.

Definition

A principal $G$-Higgs bundle is a pair $(P, \Phi)$ where

- $P$ is a holomorphic principal $H^\mathbb{C}$-bundle;
- $\Phi$ is a holomorphic section of $(P \times_{\text{Ad}} \mathfrak{m}^\mathbb{C}) \otimes K$.

See example.
**G-Higgs bundles as fixed points**

**in the fibres of the $G^c$ Hitchin fibration**

- $G$ a real form of $G^c$ fixed by the anti-holomorphic involution $\tau$
- $\rho$ the compact real form of $G^c$.
- $\sigma = \rho \tau$ a holomorphic involution.

$$\Theta : (E, \Phi) \mapsto (\sigma(E), -\sigma(\Phi))$$

N.Hitchin ‘87, ‘92
**G-Higgs bundles as fixed points**

*In the fibres of the $G^c$ Hitchin fibration*

- $G$ a real form of $G^c$ fixed by the anti-holomorphic involution $\tau$.
- $\rho$ the compact real form of $G^c$.
- $\sigma = \rho \tau$ a holomorphic involution.

$$\Theta : (E, \Phi) \mapsto (\sigma(E), -\sigma(\Phi))$$

See Example.

N. Hitchin '87, García-Prada – Gothen – Mundet ’09
**SPECTRAL DATA APPROACH I**

**Discrete intersection of \( M^{\Theta}_{G^c} \) with the smooth fibres**

\( G \)-Higgs bundles for split real forms \( (G = SL(n, \mathbb{R}), \ Sp(2n, \mathbb{R}), \ldots) \)

Theorem (*thesis*)

The intersection of \( M^{\Theta}_{G^c} \) with the smooth fibres of the Hitchin fibration

\[
 h : M_{G^c} \to \mathcal{A}_{G^c}.
\]

*is the space of elements of order 2 over the regular locus of \( \mathcal{A}_{G^c} \).*

So we can study the fibration through the monodromy action...

\( G = SL(2, \mathbb{R}) \) case (*thesis*)

What happens for other groups?
What kind of curve does $\text{char}(\Phi)$ define for a $G$-Higgs field $\Phi$?

Generically, a smooth curve for $G = U(p,p), \, SU(p,p)$...

Definition

A $U(p,p)$-Higgs bundle over $\Sigma$ is a pair $(E, \Phi)$ where $E = V \oplus W$ for $V, W$ rank $p$ vector bundles over $\Sigma$, and $\Phi$ the Higgs field given by

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

for $\beta : W \to V \otimes K$ and $\gamma : V \to W \otimes K$. When $\Lambda^p V \cong \Lambda^p W^*$, one has an $SU(p,p)$-Higgs bundle.

$$\det(x - \Phi) = x^{2p} + a_1x^{2p-2} + \ldots + a_{p-1}x^2 + a_p$$
Spectral data approach II
Towards the spectral data for $U(p,p)$-Higgs bundles

- Smooth $S$ given by $\eta^{2p} + a_1 \eta^{2p-2} + \ldots + a_{p-1} \eta^2 + a_p = 0$;
- Smooth $\tilde{S}$ given by $\xi^p + a_1 \xi^{p-1} + \ldots + a_{p-1} \xi + a_p = 0$ for $\xi = \eta^2$;
for $\eta$ tautological section of $\rho^* K$ and $a_i \in H^0(\Sigma, K^{2i})$.

Can be adapted to study $SU(p,p)$-Higgs bundles...
**SPECTRAL DATA APPROACH II**

Towards the spectral data for \(U(p, p)\)-Higgs bundles

- Smooth \(S\) given by \(\eta^{2p} + a_1 \eta^{2p-2} + \ldots + a_{p-1} \eta^2 + a_p = 0\);
- Smooth \(\bar{S}\) given by \(\xi^p + a_1 \xi^{p-1} + \ldots + a_{p-1} \xi + a_p = 0\) for \(\xi = \eta^2\);

for \(\eta\) tautological section of \(\rho^*K\) and \(a_i \in H^0(\Sigma, K^{2i})\).

- \(M := \text{coker}(\rho^*\Phi - \eta)\) line bundle on \(S\).

\[\sigma^*M \cong M\]

\[\pi_*M = U_1 \oplus U_2\]

\[\rho_*M = V \oplus W\]

\[\bar{\rho}_*U_1 = V, \quad \bar{\rho}_*U_2 = W\]

Can be adapted to study \(SU(p, p)\)-Higgs bundles...
There is a one to one correspondence between $U(p, p)$-Higgs bundles $(V \oplus W, \Phi)$ on $\Sigma$ with $\deg V > \deg W$ and non-singular spectral curve, and triples $(\bar{S}, U_1, D)$ where

- $\bar{\rho} : \bar{S} \to \Sigma$ is an irreducible non-singular $p$-cover of $\Sigma$ in the total space of $K$ with equation
  $$\xi^p + a_1 \xi^{p-1} + \ldots + a_{p-1} \xi + a_p = 0,$$
  for $a_i \in H^0(\Sigma, K^{2i})$, and $\xi$ the tautological section of $\bar{\rho}^* K^2$.
- $U_1$ is a line bundle on $\bar{S}$ whose degree is
  $$\deg U_1 = \deg V + (2p^2 - 2p)(g - 1)$$
- $D$ is a positive subdivisor of the divisor of $a_p$ of degree
  $$\tilde{m} = \deg W - \deg V + 2p(g - 1).$$
SPECTRAL DATA APPROACH II

SPECTRAL DATA FOR $U(p,p)$-Higgs bundles: the invariants

Since $\sigma^* M \cong M$ then

$$H^0(\rho^{-1}(U), M) = H^0(\rho^{-1}(U), M)^+ \oplus H^0(\rho^{-1}(U), M)^-$$

$$h^+ := \dim H^0(\rho^{-1}(U), M)^+ = \dim H^0(U, V),$$

$$h^- := \dim H^0(\rho^{-1}(U), M)^- = \dim H^0(U, W).$$

Use the $L_\sigma$ Lefschetz number [Atiyah-Bott 1968] associated to the involution $\sigma$ on $S$

$$L_\sigma = \sum (-1)^q \text{trace} \sigma|_{H^0,q(M)} = \text{trace} \sigma|_{H^0(M)} = h^+ - h^-$$

$$L_\sigma = \frac{(-\tilde{m}) + (4p(g-1) - \tilde{m})}{2} = 2p(g-1) - \tilde{m}.$$

$$\deg U_1 = v + (2p^2 - 2p)(g-1) = \frac{\deg M}{2} - \frac{\tilde{m}}{2},$$

$$\deg U_2 = w + (2p^2 - 2p)(g-1) = \frac{\deg M}{2} + \frac{\tilde{m}}{2} - 2p(g-1).$$
SPECTRAL DATA APPROACH III

No intersection of $\mathcal{M}_{G^c}$ with the smooth fibres

What kind of curve does $\text{char}(\Phi)$ define for a $G$-Higgs field $\Phi$?

Generically, a reducible curve for $G = Sp(2p, 2p), \ SU(p, q), \ldots \ (p \neq q)$

Definition

An $Sp(2p, 2p)$-Higgs bundle is a pair $(V \oplus W, \Phi)$ for $V$ and $W$ rank $2p$ symplectic vector bundles, and the Higgs field

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \text{ for } \begin{cases} \beta : W \rightarrow V \otimes K \\ \gamma : V \rightarrow W \otimes K \end{cases} \text{ and } \beta = -\gamma^T,$$

for $\gamma^T$ the symplectic transpose of $\gamma$.

$$\det(x - \Phi) = (x^{2p} + a_1x^{2p-2} + \ldots + a_{p-1}x^2 + a_p)^2$$

Note this is the case of $SU^*(2p)$ and $SO^*(2p)$, current work w/ N. Hitchin to appear soon.
SPECTRAL DATA APPROACH III
TOWARDS THE SPECTRAL DATA FOR $Sp(2p, 2p)$-HIGGS BUNDLES

- Smooth $S$ given by $\eta^{2p} + a_1\eta^{2p-2} + \ldots + a_{p-1}\eta^2 + a_p = 0$;
- Smooth $\bar{S}$ given by $\xi^p + a_1\xi^{p-1} + \ldots + a_{p-1}\xi + a_p = 0$ for $\xi = \eta^2$;

for $\eta$ tautological section of $\rho^*K$ and $a_i \in H^0(\Sigma, K^{2i})$.

$$\sigma:\eta \rightarrow -\eta$$

$$S \xrightarrow{2:1} \bar{S} = S/\sigma$$

$$\rho \xrightarrow{p:1} \bar{\rho}$$
SPECTRAL DATA APPROACH III
Towards the spectral data for $Sp(2p, 2p)$-Higgs bundles

- Smooth $S$ given by $\eta^{2p} + a_1 \eta^{2p-2} + \ldots + a_{p-1} \eta^2 + a_p = 0$;
- Smooth $\tilde{S}$ given by $\xi^p + a_1 \xi^{p-1} + \ldots + a_{p-1} \xi + a_p = 0$ for $\xi = \eta^2$;

for $\eta$ tautological section of $\rho^*K$ and $a_i \in H^0(\Sigma, K^{2i})$.

$M := \text{coker}(\rho^*\Phi - \eta)$ rank 2 vector bundle on $S$.

$\sigma^*M \cong M$

$\sigma : \eta \rightarrow -\eta$

$\begin{array}{ccc}
S & \xrightarrow{\pi} & \tilde{S} = S/\sigma \\
2:1 & & 2p:1 \\
\rho & & p:1 \\
\rho^*M = V \oplus W & \rightarrow & \Sigma \\
\end{array}$
Spectral data approach III
Spectral data for $Sp(2p, 2p)$-Higgs bundles (thesis)

There is a one to one correspondence between stable $Sp(2p, 2p)$-Higgs bundle $(E = V \oplus W, \Phi)$ on $\Sigma$ for which $\text{char}(\Phi)^{1/2} = 0$ defines a smooth curve, and the spectral data $(S, M)$ where

(a) the curve $\rho : S \to \Sigma$ is a smooth $2p$-fold cover with equation

$$\eta^{2p} + a_1 \eta^{2p-2} + \ldots + a_{p-1} \eta^2 + a_p = 0,$$

in the total space of $K$, where $a_i \in H^0(\Sigma, K^{2i})$, and $\eta$ is the tautological section of $\rho^* K$. The curve $S$ has a natural involution $\sigma$ acting by $\eta \mapsto -\eta$;

(b) $M$ is a rank 2 vector bundle on $S$ with $\Lambda^2 M \cong \rho^* K^{-2p+1}$, and such that $\sigma^* M \cong M$. Over the fixed points of the involution, the vector bundle $M$ is acted on by $\sigma$ with eigenvalues $+1$ and $-1$. 
**APPLICATIONS**

**Connectivity for \( M_{U(p,p)} \)**

\( U(p, p) \)-Higgs bundle of fixed degree \( \sim (\bar{S}, U_1, D) \) with fixed deg \( M \).

- The choice of \( D \) lies in the symmetric product \( S^{\tilde{m}}\Sigma \);
- Together with a section \( s \) of \( K^{2p}[-D] \) with distinct zeros, \( D \) gives the map \( a_p \in H^0(\Sigma, K^{2p}) \);
- The choice of \( a_p \) lies in a vector bundle of rank \((4p - 1)(g - 1) - \tilde{m}\) over \( S^{\tilde{m}}\Sigma \), whose total space is \( E \); There is a natural map

\[
\alpha : E \rightarrow H^0(\Sigma, K^{2p})
\]

- The choice of \( \bar{S} \) is given by a point in a Zariski open \( A \) in

\[
H^0(\Sigma, K^{2p}) \oplus \bigoplus_{i=1}^{p-1} H^0(\Sigma, K^{2i})
\]

- The choice of \( U_1 \) is given by a fibration of Jacobians \( Jac \) over \( A \);
APPLICATIONS

Connectivity for $\mathcal{M}_{U(p,p)}$ (thesis)

Each pair of invariants $(m, \tilde{m})$ labels exactly one connected component of $\mathcal{M}_{U(p,p)}$ which intersects the non-singular fibres of the Hitchin fibration

$$\mathcal{M}_{GL(2p,\mathbb{C})} \rightarrow \mathcal{A}_{GL(2p,\mathbb{C})}.$$ 

This component is given by the fibration of $\alpha^* \mathcal{J}ac$ over a Zariski open subset in

$$\mathcal{E} \bigoplus_{i=1}^{p-1} H^0(\Sigma, K^{2i}).$$

$\mathcal{M}_{U(p,q)}$ via Morse theory by Bradlow–García-Prada–Gothen ‘02
Applications
Connectivity for $\mathcal{M}_{Sp(2p,2p)}$

$Sp(2p,2p)$-Higgs bundles with smooth spectral curve $\sim (S,M)$ for $M$ rank 2 vector bundle with $\Lambda^2 M \cong \rho^* K^{-2p+1}$ and conditions on $\sigma^* M \cong M$. 

\[ \sigma^* M \cong M \]
\[ \sigma : \eta \rightarrow -\eta \]
\[ S \xrightarrow{\pi} \bar{S} = S/\sigma \]
\[ \rho \]
\[ \rho_* M = V \oplus W \]
\[ \Sigma \]
Applications
Connectivity for $\mathcal{M}_{Sp(2p,2p)}$

$Sp(2p,2p)$-Higgs bundles with smooth spectral curve $\sim (S, M)$ for $M$ rank 2 vector bundle with $\Lambda^2 M \cong \rho^* K^{-2p+1}$ and conditions on $\sigma^* M \cong M$.

\[
\begin{align*}
\sigma^* M &\cong M \\
\rho^* M &= V \oplus W \\
\bar{\rho} &\rightarrow \left\{ \begin{array}{c} 2p:1 \\ p:1 \end{array} \right. \\
\pi &\rightarrow \bar{S} = S/\sigma
\end{align*}
\]
Applications

Connectivity for $\mathcal{M}_{Sp(2p,2p)}$

$Sp(2p,2p)$-Higgs bundles with smooth spectral curve $\sim (S,M)$ for $M$ rank 2 vector bundle with $\Lambda^2M \cong \rho^*K^{-2p+1}$ and conditions on $\sigma^*M \cong M$.

- $N^\sigma =$ fixed point set of $\sigma$ in the moduli space of stable rank 2 vector bundles of determinant $\rho^*K^{2p-1}$;
- $P_a$ the moduli space parabolic rank 2 vector bundles on $\bar{S}$ whose marked points are the fixed points of the involution $\sigma$, whose weights are $1/2$ and whose flag is by the distinguished eigenspaces corresponding to the eigenvalue $-1$ of $\sigma$ [Andersen-Grove 2006];
  - Vector bundles in $P_a$ are stable [Nitsure 86];
  - $P_a = P_a^+ \sqcup P_a^c$ through a natural involution on $P_a$;
  - $P_a^c$ is connected [Nitsure 86];

The choice of $M$ is given by an element in

$$N^\sigma \cong P_a^c$$
The space $\mathcal{M}_{Sp(2p,2p)}^s$ is given by the fibration of $\mathcal{P}_a^c$, over a Zariski open set in the space

$$\bigoplus_{i=1}^{p} H^0(\Sigma, K^{2i}).$$

$\mathcal{M}_{Sp(2p,2q)}$ via Morse Theory by García-Prada–Oliveira ‘12
**Applications**

**Topological invariants**

Milnor-Wood type inequalities for the Toledo invariant $\tau(v, w)$ associated to $G$-Higgs bundles appear naturally from the spectral data...

- $U(p, p)$-Higgs bundles, for which $\tau(v, w) = v - w$
  - The invariant $\tilde{m} = w - v + 2p(g - 1)$ is the number of fixed points of $\sigma$ with certain property.
  - Fixed points of $\sigma$ are zeros of $a_p \in H^0(\Sigma, K^{2p})$, thus
    $$0 \leq w - v + 2p(g - 1) \leq 4p(g - 1)$$
    $$|\tau(v, w)| = |v - w| \leq 2p(g - 1)$$

- $SU(p, p)$-Higgs bundles, for which $\tau(v, w) = v = -w$
  - Methods for $U(p, p)$ can be adapted, and we get
    $$|\tau(v, w)| = |v| \leq p(g - 1)$$

- Also for $Sp(2n, \mathbb{R})$, $Sp(2p, 2p)$... and possibly others?
Thank you for listening!