K-homology and elliptic operators

Let X be a compact manifold.
D an elliptic 1st order operator on X.
Then D is a Fredholm operator and so it has an index.
If $E \to X$ is a vector bundle, then a standard construction extends D to an elliptic operator D_E acting on sections of $E \to X$. The assignment

$E \to \mathrm{Index}\, D_E$ determines a morphism $\mathrm{Index}_D : K^0(X) \to \mathbb{Z}$.

So Atiyah concluded that $(D, L^2(X))$ should represent a K-homology class under suitable equivalence relations, which Kasparov determined and developed. K-homology conceptualizes many constructions in index theory.

eg The index is induced from the map $X \to pt$.

Let A be a C^*-algebra. A <u>Fredholm A-module</u> is

1) a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$,
2) a (grading-preserving) $*$ homomorphism
$$\phi : A \longrightarrow B(\mathcal{H})$$
3) A self adjoint operator $T \in B(\mathcal{H})$ of the form $T = T^* = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$ satisfying
   a) $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ (ellipticity) $\forall a \in A$
   b) $[\phi(a), T] \in \mathcal{K}(\mathcal{H})$ $\forall a \in A$ (pseudolocal)

There is a natural associative <u>addition</u> on Fredholm A-modules by taking the direct sum.

The equivalence relation is generated by the addition of <u>degenerate</u>

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Fredholm A-modules (for which $\phi(a)(T^2-1)=0$ & $[\phi(a),T]=0 \ \forall a \in A$) and also by <u>homotopy</u>. Then if $K^0(A)$ denotes the equivalence classes of Fredholm A-modules, then $K^0(A)$ is an Abelian group, with the inverse given by reversing the grading ie $\mathcal{H}^0 \to \mathcal{H}^1$ & $\mathcal{H}^1 \to \mathcal{H}^0$, and interchanging F and F^*.

Given a finite projective A-module, it is of the form $e(A^n)$ for some projection $e \in M_n(A)$.

Also given $(\mathcal{H}, F, \phi)$ a Fredholm A-module, we get $(\mathcal{H} \otimes \mathbb{C}^n, F \otimes I_n, M_n(\phi) = \phi_n)$ is a Fredholm $M_n(A)$-module

$(F \otimes I_n)\phi_n(e)$ acting on $\phi_n(e)(\mathcal{H} \otimes \mathbb{C}^n)$ is a Fredholm operator and index $(\varphi_n(e)(F\otimes I_n)\varphi_n(e)) \equiv \langle F, e\rangle$ depends only on $[F]$ in $K^0(A)$ and $[e]$ in $K_0(A)$, giving rise to a bilinear map $K^0(A)\times K_0(A)\to \mathbb{Z}$.

When $A = C(X)$, then $K_0(X) = K^0(C(X))$ is the K-homology group.
D a 1st order elliptic operator on X $\Rightarrow F = \chi(D)$ is a bounded Fredholm operator where $\chi\in C(\mathbb{R})$, $\chi(x) = \pm 1$ as $x\to\pm\infty$ is a normalizing function eg $\chi(x) = \frac{x}{\sqrt{1+x^2}}$ and we get a Fredholm $C(X)$-module.

In the special case when:
G is noncompact and $A = C^*(G)$,
A Fredholm module over G consists of a pair of Hilbert space representations V_0, V_1 of G, together with
1) $U : V_0 \to V_1$ a bdd operator, which is unitary modulo compact operators

(equivariant)
2) It is an intertwiner modulo compact operators $gU - Ug \in \mathcal{K}(\mathcal{H}_0 \oplus \mathcal{H}_1)$

These are the Fredholm $C^*(G)$-modules and $K^0(C^*(G))$ is sometimes denoted $R_{Fred}(G)$. When G is compact

$$K^0(C^*(G)) \cong R(G) \cong K_0(C^*(G))$$

When $G = \mathbb{T}^n$, then

$$R(G) \cong \mathbb{Z}[\hat{G}] = \text{group ring of } \mathbb{Z}^n.$$

If G is a compact Lie group then $R(G) = R(T)^W = \mathbb{Z}[\hat{T}]^W$ where T is the maximal abelian subgroup of G

eg $SU(2)$, $\underline{R(SU(2)) = \mathbb{Z}[\mathbb{Z}]^{\mathbb{Z}_2} \cong \mathbb{Z}[\mathbb{N}]}$

~~$K_G^0(A)$ consists of G-continuous Fredholm A-modules (rather than only G-equivariant Fredholm modules). Then $K_G^0(A)$ are modules over $K_G^0(\mathbb{C}) = K^0(C^*(G)) = R_{Fred}(G)$~~

This is unknown for most connected semi-simple Lie groups.

Let $D: C^\infty(X, E_0) \to C^\infty(X, E_1^*)$ be a 1^{st} order elliptic operator. Define a morphism $S_D: T^*X \otimes E_0 \to E_1^*$ by

$$S_D(df \otimes e_0) = D(fe_0) - f De_0 \quad \forall f \in C^\infty(X)$$

and $e \in C^\infty(X, E_0)$. Since D is 1^{st} order S_D is ~~a~~ tensorial. Let ∇^E be a unitary connection on $E \to X$ ie $\nabla^E: C^\infty(X, E) \to C^\infty(X, T^*X \otimes E)$

Consider $S_D^E: E_0 \otimes (E \otimes T^*X) \to E_1 \otimes E$ by

$$S_D^E(e_0 \otimes (e \otimes df)) = S_D(df \otimes e_0) \otimes e$$

ie $C^\infty(X, E)$ linear. Then

$$D \otimes \nabla^E : C^\infty(X, E_0 \otimes E) \to C^\infty(X, E_1 \otimes E)$$

$$\boxed{D \otimes \nabla^E\Big(\sum_i e_0^i \otimes e^i\Big) = \sum_i D(e_0^i) \otimes e^i + S_D^E\Big(\sum_i e_0^i \otimes \nabla^E e^i\Big)}$$

since the space of connections on E is an affine space, the operators $D \otimes \nabla^E$ are homotopic & $[D \otimes \nabla^E] = [D_E] \in K_0(X)$

properties of K-homology

1) If $f: A \to B$ is a morphism of C^* algebras, then $f^*: K^0(B) \longrightarrow K^0(A)$ is a morphism of K-homology groups

2) If X is a cpt even dimensional (eg spin manifolds) spin$^{\mathbb{C}}$-manifold, then it has a spin$^{\mathbb{C}}$ Dirac operator $\not{D}$. The class of $[\not{D}] \in K_0(X)$ is called a <u>fundamental</u> <u>class</u> of X and X is called <u>K-oriented</u>. What this means is that the map, $(E \to X) \longrightarrow \not{D}_E$ which determines a map $K^0(X) \to K_0(X)$, is an <u>isomorphism</u> of abelian groups, called Poincaré duality.

3) In general, if X is just a compact manifold, then $K_0(X) \longrightarrow K_c^0(T^*X)$ given by $[D, L^2(E)] \to (E, \sigma(D))$ where $\sigma(D)$ is the principal symbol, is an isomorphism

Calculations

6-term exact sequence: Consider the exact sequence of C^*-algebras

$$0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0.$$

It gives rise to a 6-term exact sequence

$$\begin{tikzcd}
K^1(C) \arrow[r, "q^*"] & K^1(B) \arrow[r, "i^*"] & K^1(A) \arrow[d, "\delta"] \\
K^0(A) \arrow[u, "\delta"] & K^0(B) \arrow[l] & K^0(C) \arrow[l, "q^*"']
\end{tikzcd}$$

Example

Geometric K-homology

In the commutative case there are other descriptions of K-homology.
Since every extraordinary (co)homology theory is a quotient of the (co) bordism group, this motivates the following.

<u>Defn</u> a geometric K-cycle on X is a triple (M, E, f) where

- M is a compact spin^c manifold
- E is a Hermitian vector bundle on M
- $f : M \longrightarrow X$ is continuous.

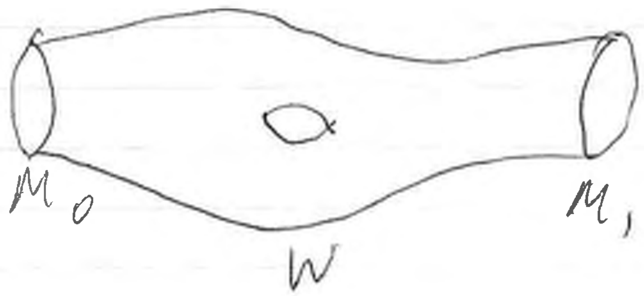
<u>NB</u> M does not have to be connected, and components of M can have different dimensions. (M, E, f) is isomorphic to (M', E', f') if there is a diffeomorphism h

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "f"'] & M' \arrow[d, "f'"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

preserving the spin^c structures and $h^*(E') \cong E$ as Hermitian vector bundles

The disjoint union $(M_1, E_1, f_1) \amalg (M_2, E_2, f_2)$ is the addition. Let $\Pi(X)$ denote the space of all geometric K-cycles upto isomorphism. Define the equivalence relation $\sim$ on $\Pi(X)$ generated by

1) Bordism $(M_0, E_0, f_0) \sim (M_1, E_1, f_1)$ if there is a compact mfd with boundary W



a vector bundle $E \to W$ and a cts map $f: W \to X$ such that

$(\partial W, E|_{\partial W}, f|_{\partial W})$ is isomorphic to the disjt union (M_0, E_0, f_0) and $(-M_1, E_1, f_1)$ where $-M_1 = M_1$ with opposite spin^c structure.

2) Direct sum $(M, E_0 \oplus E_1, f) \sim (M, E_0, f) \amalg (M, E_1, f)$

3) Vector bundle modification

Let (M, E, f) be a geometric K-cycle and $H \to M$ be an even dimensional spin$^{\mathbb{C}}$ vector bundle. Then $H \oplus 1_{\mathbb{R}}$ is an odd dimensional vector bundle on M $(1_{\mathbb{R}} = M \times \mathbb{R})$ and $\hat{M} = S(H \oplus 1_{\mathbb{R}})$ is an even dimensional sphere bundle over M $\pi : \hat{M} \to M$ the projection. Then $\hat{S} = \pi^*(S)$ is a spin$^{\mathbb{C}}$ structure for $\hat{M}$, where S is the spinor bundle of H

Also $\hat{S} = \hat{S}_0 \oplus \hat{S}_1$, define $\hat{H} = \hat{S}_0^*$

Define $\hat{E} \to \hat{M}$ by $\hat{E} = \hat{H} \otimes \pi^* E$ and $\hat{f} = f \circ \pi$. Then $(\hat{M}, \hat{E}, \hat{f})$ is a geometric K-cycle for X. Then

$(\hat{M}, \hat{E}, \hat{f}) \sim (M, E, f)$

Let $\Pi^{\text{even}}(X) =$ even dimensional geometric K-cycles for X,

$\Pi^{\text{odd}}(X) =$ odd dimensional geometric K-cycles for X.

$$K_0^{geom}(X) = \Pi^{even}(X)/\sim$$

$$K_1^{geom}(X) = \Pi^{odd}(X)/\sim$$

The map $K_\bullet^{geom}(X) \longrightarrow K_\bullet(X)$

$$(M, E, f) \longrightarrow (f_*(\not{D}_E))$$

is an <u>isomorphism</u> (Baum-Higson-Schick)

What about odd K-homology?

One can define $K^d(A) = K^0(A \otimes C_0(\mathbb{R}^d))$. Then by Bott periodicity, $K^d(A) \cong K^{d+2}(A)$ and therefore there are just two K-homology groups $K^0(A)$ and $K^1(A)$.

There are alternate descriptions of $K^1(A)$ that are sometimes more useful. An odd Fredholm A-module is a Fredholm A-module with a trivially graded Hilbert space. Let $(\mathcal{H}, F, \varphi)$ be an odd Fredholm A-module and $\psi \in GL_N(A)$. Since F is self-adjoint and Fredholm, let $\chi_+(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$. Then $F \otimes I_N$ is also sa Fredholm on $\mathcal{H} \otimes \mathbb{C}^N$ and $\chi_+(F \otimes I_N)$ are projections, $F_\psi = \chi_+(F \otimes I_N) M_\psi \chi_+(F \otimes I_N)$ where M_ψ is the multiplication operator $\varphi(\psi)$ acting on $\mathcal{H} \otimes \mathbb{C}^N$. F_ψ is a Fredholm operator and induces a pairing

$$K^1(A) \times K_1(A) \longrightarrow \mathbb{Z}$$

$$((\mathcal{H}, F, \varphi) \times \psi) \longrightarrow \text{index}(F_\psi)$$

F_ψ is a Toeplitz operator.

Examples Let D be a self-adjoint elliptic operator on X a compact manifold, then $[D] \in K_1(X)$. In particular, if X is a spin$^{\mathbb{C}}$-manifold of odd dimension. Then the spin$^{\mathbb{C}}$ Dirac operator $\not{D}$ is a self-adjoint elliptic operator, $[\not{D}] \in K_1(X)$.

More generally, if $E \to X$ is a vector bundle, then $\not{D}_E$ is also a self-adjoint elliptic operator and the map

$$K^0(X) \longrightarrow K_1(X)$$

$$E \longrightarrow [\not{D}_E]$$

is an isomorphism, called Poincaré duality

Extensions and K-homology

Let $\mathcal{K}=\mathcal{K}(\mathcal{H})$ be compact operators on a Hilbert space and the ses

(1) $0 \to \mathcal{K} \to B \to A \to 0$

be an <u>extension of A by $\mathcal{K}$</u>. That is, B is a C^*-algebra, $\mathcal{K} \hookrightarrow B$ a 2-sided ideal in B.

Example

(2) $0 \to \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}) \to \underset{\substack{\parallel \\ Q(\mathcal{H})}}{B(\mathcal{H})/\mathcal{K}(\mathcal{H})} \to 0$

is an extension of the Calkin algebra by $\mathcal{K}$.

Given (1), use (2) to get $\tau : A \to Q(\mathcal{H})$ as follows:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K}(\mathcal{H}) & \longrightarrow & B & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \| & & \downarrow i & & \downarrow \tau & & \\ 0 & \to & \mathcal{K}(\mathcal{H}) & \longrightarrow & B(\mathcal{H}) & \xrightarrow[q]{} & Q(\mathcal{H}) & \longrightarrow & 0 \end{array}$$

Given $a \in A$, choose $b \in B$ with $p(b)=a$. Define $\tau(a) = q(i(b))$. τ is called the Busby invariant of the extension (1)

Conversely, given $\tau : A \longrightarrow Q(\mathcal{H})$ a morphism (Busby invariant), define

$$B_\tau = \{ (T, a) \in B(\mathcal{H}) \oplus A \mid q(T) = \tau(a) \}$$

then we have an extension

$$
\begin{array}{ccccccccc}
0 \to & K & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
 & & & (T, a) & \longrightarrow & a & & \\
 & k & \longrightarrow & (k, 0) & & & &
\end{array}
$$

Two morphisms $\tau_1, \tau_2 : A \to Q(\mathcal{H})$ are unitarily equivalent if there is a unitary U such that $\tau_1(a) = U\tau_2(a)U^*$ $\forall\, a \in A$

Addition is direct sum $\tau_1 \oplus \tau_2 : A \to M_2(Q(\mathcal{H})) \subset Q(\mathcal{H} \otimes \mathbb{C}^2)$. A Busby invariant $\tau : A \to Q(\mathcal{H})$ is trivial if the corresponding $\mathcal{K}$-extension is split $0 \to K \to B_\tau \rightleftarrows A \to 0$

A Busby invariant $\tau : A \to Q(\mathcal{H})$ is invertible if $\exists$ $\tau_1 : A \to Q(\mathcal{H})$ such that $\tau \oplus \tau_1$ is trivial

$Ext(A)$ = the group of invertible K-extensions of A, modulo unitary equivalence and split extensions

Thm $Ext(A) \cong K^1(A)$

given an <u>odd</u> Fredholm A-module $(\mathcal{H}, F, \varphi)$, $P = \chi_+(F)$, define $\tau : A \to \mathcal{Q}(\mathcal{H})$

$\tau(a) = q\left(P\left(\varphi(a)\cancel{E}\right)P\right)$ is a Busby invariant defining the extension

$$0 \to \mathcal{K} \to B_\tau \to A \to 0$$

Equivariant K-homology

Let G be a locally compact Lie group. A G-C^*-algebra is a C^*-algebra A with jointly continuous action of G on A ie $G \longrightarrow Aut(A)$ is continuous. If G is compact, then making K-homology equivariant is straightforward. One just requires all algebras and Hilbert spaces ~~to~~ have G-actions, $\varphi : A \to B(\mathcal{H})$ to be G-equivariant. Then the equivariant K-homology, $K_G^{\bullet}(A)$ is an abelian group, which is a module over $K_G^0(\mathbb{C}) = R(G)$ the representation ring and everything works nicely in this case

When G is noncompact, the definition and properties of equivariant K-homology are much more subtle. The problem is that topological vector spaces with a cts G-action are rarely decomposable and there are rarely enough G-equivariant operators. Therefore, instead of G-equivariant Hilbert spaces and operators, one works with <u>G-continuous</u> Hilbert spaces and remarkably, these give a useful theory with the same formal properties reducing to the equivariant definition in the compact case, see [Kasparov]. $K_G^{\bullet}(A)$ are again modules over $K_G^0(\mathbb{C}) = K^0(C^*(G)) = R_{fred}(G)$ which is much less understood for G noncompact

Functorial properties

Restriction Let A be a G-C^*-algebra and H a subgroup of G. Then A is also a H-C^*-algebra and we have

$$\mathrm{Res}_H^G : K_G^\bullet(A) \longrightarrow K_H^\bullet(A).$$

Induction Given an H-C^*-algebra B, we can define a G-C^*-algebra

$$\mathrm{Ind}_H^G(B) = \{ f \in C(G, A) \mid f(gh) = h\, f(g),\ g \in G, h \in H \text{ and } \|f(g)\| \to 0 \text{ as } [g] \to \infty \text{ in } G/H \}$$

The action of G on $\mathrm{Ind}_H^G(B)$ is by left translation, and we have

$$\mathrm{Ind}_H^G : K_H^\bullet(B) \longrightarrow K_G^\bullet(\mathrm{Ind}_H^G(B))$$

Crossed product algebras

Given a G-C^*-algebra A, one can form 2 new C^*-algebras called the full and reduced crossed product algebras. The full crossed product $A \rtimes G$ is defined as the universal C^*-algebra of covariant pairs (φ, π) where φ is a $*$-representation of A on $\mathcal{H}$, and π is a representation of G on $\mathcal{H}$, satisfying the compatibility

$$\pi(g)\varphi(a)\pi(g^{-1}) = \varphi(\alpha_g(a)) \quad \forall g \in G, a \in A$$

To construct it, consider $C_c(G, A)$ where $f_1 * f_2(g) = \int_G f_1(h)\, \alpha_h(f_2(h^{-1}g))\, dh$ for $f_1, f_2 \in C_c(G, A)$. Then $A \rtimes G = \overline{C_c(G, A)}$ in the greatest C^*-norm which is estimated above by $\|f\|_{L^1}$.

The reduced crossed product $A \rtimes_r G$ consists of $C_c(G, A)$ acting on $L^2(G, \mathcal{H})$

where $\mathcal{H}$ is a Hilbert space on which A acts faithfully via φ. Then A acts by $(\pi(a)f)(g) = \varphi(\alpha_{g^{-1}}(a))f(g)$ for $f \in L^2(G,\mathcal{H})$, and G acts by left translations.

<u>Examples</u> 1) If $A = \mathbb{C}$ with the trivial G-action, then $\mathbb{C} \rtimes G = C^*(G)$ and $\mathbb{C} \rtimes_r G = C^*_r(G)$

2) More generally, if A is a G-C^*-algebra with the <u>trivial</u> action of G, then

$A \rtimes G = A \otimes_{max} C^*(G)$ whereas

$A \rtimes_r G = A \otimes_{min} C^*_r(G)$.

3) If X is a locally compact G-space where G acts <u>properly</u>, then $C_0(X) \rtimes_r G \cong C_0(X) \rtimes G$ for all locally compact Lie groups G.

Why study crossed product algebras?

When X is a locally compact G-space the crossed product $C_0(X) \rtimes G$ is a desingularization of X/G. Indeed

- if G acts properly and freely on X, then $C_0(X) \rtimes G$ and $C_0(X/G)$ are Morita equivalent.
- If G acts properly and <u>locally</u> freely on X, then X/G is an orbifold and one takes its space of "functions" to be $C_0(X) \rtimes G$.
- If the G action is <u>not</u> proper, then X/G may be non-Hausdorff and $C_0(X) \rtimes G$ is a nice C^*-algebra of "functions".
- $X = S^1$, $G = \mathbb{Z}$, the generator of $\mathbb{Z}$ acts by multiplication by $e^{2\pi i \theta}$. When θ is irrational, then every G-orbit is dense in S^1 & $S^1/\mathbb{Z}$ is non Hausdorff

The crossed product $C(S^1) \rtimes \mathbb{Z}$, usually denoted A_θ, is the noncommutative torus.

Green-Rieffel theorems

1) Let G be a Lie group, H and K be closed subgroups. Then

$$C_0({}_H\backslash G) \rtimes K \quad \text{and} \quad C_0(G/K) \rtimes H$$

are Morita equivalent.

2) Let A be a H-C^*-algebra and H a closed subgroup of G. Then

$$\mathrm{Ind}_H^G(A) \rtimes G \quad \text{and} \quad A \rtimes H$$

are Morita equivalent.

K-homology (and K-theory) are Morita invariant.

Connes-Thom isomorphism

Let A be a $\mathbb{R}$-C^*-algebra. Then there is a natural isomorphism

$$K^\bullet(A) \longrightarrow K^{\bullet+1}(A \rtimes \mathbb{R})$$