Twisted K-Theory and Families Index Problem on Product Manifolds

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Introduction

- Let X be a compact manifold.
  Consider ordinary K-theory given by the groups

$$K^0(X) = [X, \mathbf{Fred}^{(0)}(\mathcal{H})] \qquad K^1(X) = [X, \mathbf{Fred}^{(1)}(\mathcal{H})]$$

  - $\mathcal{H}$ is a complex ∞-dim Hilbert space,
  - $\mathbf{Fred}^{(0)}$ the space of bounded Fredholm operators,
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  - $\mathbf{Fred}^{(0)}$ the space of bounded Fredholm operators,
  - $\mathbf{Fred}^{(1)}$ the space of bounded self-adjoint Fredholm operators.
- Many interesting examples of Fredholm operators are unbounded
  - for example elliptic differential operators
  - the usual strategy is to realize the K-theory classes as approximated signs of such families

$$\frac{\eth}{\sqrt{1+(\eth)^2}}.$$

Introduction - the case of a torus

- Let $X = \mathbb{T}^n$ be the base of the fibration

$$\mathbb{T}^2 \hookrightarrow \mathbb{T}^2 \times \mathbb{T}^n \to \mathbb{T}^n$$

Fix a vector bundle ξ over $\mathbb{T}^2 \times \mathbb{T}^n$ and couple the Dirac family on the fibres $\mathbb{T}^2$ to the potential of ξ. Such an operator acts on $\mathbb{Z}_2$-graded bundle of smooth spinors as

$$\begin{pmatrix} 0 & \eth_\xi^* \\ \eth_\xi & 0 \end{pmatrix}.$$

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The characteristic class of the (stablilized) index

$$\mathrm{ind}(\eth_\xi) = \ker(\eth_\xi) - \mathrm{coker}(\eth_\xi)$$

in de Rham cohomology is the form

$$\textbf{ch-ind}(\eth_\xi) = \frac{1}{2\pi i}\varphi \int_{\mathbb{T}^2} \mathrm{ch}(\xi).$$

- the cohomology class **ch-ind** is invariant under smooth homotopies of the K-theory representative of $\eth$.

- Consider the fibration with odd dimensional fibres

$$\mathbb{T}_0 \hookrightarrow \mathbb{T}_0 \times \mathbb{T}^n \to \mathbb{T}^n$$

Couple the family of Dirac operators on the fibres $\mathbb{T}_0$ to the potential of the line bundle with the curvature

$$\frac{i}{2\pi} d\phi_0 \wedge d\phi_1.$$

This gives a Dirac family of the form

$$\eth = -i\partial_{\phi_0} + \phi_1.$$

- no $\mathbb{Z}_2$-graded spinors.
- spectral flow around ϕ_1.

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- Twist the spinor bundle by tensoring with a complex vector bundle

$$\xi \to \mathbb{T}_2 \times \cdots \times \mathbb{T}_n.$$

- The odd K-theory $K^1(\mathbb{T}^n)$ can be identified with a subgroup in $K^0(\mathbb{T} \times \mathbb{T}^n)$ with virtual dimension zero (the rank of the index bundle) which vanish in $* \times M$.
  - a representative of such map is given by the Atiyah-Singer suspension

$$\mathrm{susp} : K^1(\mathbb{T}^n) \to K^0(\mathbb{T}^{n+1}).$$

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- Define the Chern character of $\eth_\xi$ to be the composition

$$\begin{aligned} \textbf{ch-ind}_1 \quad &: \quad \Psi^{-1} \circ \textbf{ch-ind} \circ \text{susp}(\eth_\xi) \\ &: \quad K^1(\mathbb{T}^n) \to H^{\text{odd}}(\mathbb{T}^n). \end{aligned}$$

  where Ψ^{-1} is the desuspension in cohomology:

$$\Psi^{-1} : H^{\text{even}}(\mathbb{T}^{n+1}) \to H^{\text{odd}}(\mathbb{T}^n) \qquad \Psi^{-1} = \frac{i}{2\pi} \int_{\mathbb{T}_s}$$

- A Dirac suspension is an even Dirac family over $\mathbb{T}^{n+1}$ which is homotopic to the Atiyah-Singer suspension

$$[\eth^s_\xi] = [\mathsf{susp} \circ \eth_\xi]$$

- the character **ch-ind**$(\eth^s_\xi)$ can be solved with the families index formula.

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The character **ch-ind**$_1$ of $\eth_\xi$ is equal to

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- The 1-form part computes the spectral flow around ϕ_1
The 3-form part is an obstruction of quantization
- Carey-Mickelsson-Murray.

Twisted K-Theory

- Let X be a compact manifold with a good cover $\{V_i\}$.
  We consider a K-theory twisted by a representative of $H^2(X, \underline{\mathbb{T}})$
  - such class is determined by locally defined line bundles

$$\{\lambda_{ij} \to V_{ij} = V_i \cap V_j\}$$

  - the components of the Cech cocycles are local bundle isomorphism

$$\{f_{ijk} : V_{ijk} \to \mathbb{T}; \qquad \lambda_{ij} \otimes \lambda_{jk} \to \lambda_{ik}\}.$$

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$$u_{ij} : \mathbf{H}_j \to \mathbf{H}_i \otimes \lambda_{ij}.$$

- Let the unitary transformations u_{ij} act on the spaces of Fredholm operators $\underline{\mathbf{Fred}}^{(\bullet)}$ by conjugation. Then the bundles of Freholm operators

$$\{\underline{\mathbf{Fred}}^{(\bullet)}(\mathbf{H}_i)\}$$

  glue together to form a fibre bundle.
  - phase factors vanish under conjugation action.

A K-theory twisted by the repersentative $\{\lambda_{ij}\}$ in $H^2(X, \underline{\mathbb{T}})$ is defined by

$$K^{\bullet}(X, f) = [\Gamma(\underline{\mathbf{Fred}}^{(\bullet)}(\mathbf{H}))] \qquad \bullet = 0, 1.$$

- continuous sections,
- homotopy classes in the space of such sections.

Twisted K-Theory on $\mathbb{T} \times M$

- Let M be a compact manifold.
  Fix a decomposable class

$$\tau = \alpha \smile \beta \in H^1(\mathbb{T}, \mathbb{Z}) \otimes H^2(M, \mathbb{Z}) \subset H^3(\mathbb{T} \times M, \mathbb{Z}).$$

  Fix a complex line bundle λ associated with β.

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  The twisted K-group on a product, $K^\bullet(\mathbb{T} \times M, \tau)$, is isomorphic to an extension of

$$\{x \in K^\bullet(M) : x \otimes \lambda = x\} \quad \text{by} \quad \frac{K^{\bullet+1}(M)}{(1-\lambda) \otimes K^{\bullet+1}(M)}$$

  (Solve this using the Mayer-Vietoris sequence in twisted K).

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  Take $M = \mathbb{T}^2$, and $\beta = k\times$ the generator of $H^2(M)$, then

$$K^1(\mathbb{T}^3, \tau) = (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}_k).$$

Some History

Goals
- representatives for the elements of $K^\bullet(\mathbb{T} \times M, \tau)$
- index maps valued in a twisted cohomology theory.

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- representatives for the elements of $K^\bullet(\mathbb{T} \times M, \tau)$
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- Twisted cohomology and Chern character maps
  - Bouwknegt-Carey-Mathai-Murray-Stevenson 02

  Index for twisted Dirac families in the case of torsion twisting
  - Mathai-Melrose-Singer 05

  Index for twisted Dirac families in the decomposable case
  - Mathai-Melrose-Singer 09

  Index for twisted Dirac families - superconnection proof
  - Benameur-Gorokhovsky 11

  K^1-Index for supercharge families in the decomposable case
  - Harju-Mickelsson 12

  Extension to equivariant K^1-case with supercharge families
  - Harju 12

Gerbes on $\mathbb{T} \times M$

- Fix an open cover $\{\mathbb{T}_\downarrow \times M, \mathbb{T}_\uparrow \times M\}$ for $\mathbb{T} \times M$.
  Denote by $(\mathbb{T}_\downarrow \cap \mathbb{T}_\uparrow)^{(\pm 1)}$ the subset which contains $\pm 1 \in \mathbb{T}$.
  Fix a good open cover $\{U_a\}$ for M.

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- A gerbe associated with a decomposable class $\alpha \smile \beta$ can be viewed as a pair of locally defined Hilbert bundles $\mathbf{F}_{\downarrow\uparrow} \to \mathbb{T}_{\downarrow\uparrow} \times M$ together with isomorphisms

$$u_a : \mathbf{F}_\uparrow|_a \to \mathbf{F}_\downarrow \otimes \lambda|_a \quad \text{over } (\mathbb{T}_\downarrow \cap \mathbb{T}_\uparrow)^{(1)} \times U_a$$

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- The curvature forms of the local Hilbert bundles verify

$$u_a^*(F_\uparrow) = F_\downarrow + F_\lambda,$$

- $F_{\downarrow\uparrow}$ and F_λ are the curvature forms in $\mathbf{F}_{\downarrow\uparrow}$ and λ.

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- The curvature forms of the local Hilbert bundles verify

$$u_a^*(F_\uparrow) = F_\downarrow + F_\lambda,$$

  - $F_{\downarrow\uparrow}$ and F_λ are the curvature forms in $\mathbf{F}_{\downarrow\uparrow}$ and λ.
- There are forms $\Omega_{\downarrow\uparrow}$ defined over $\mathbb{T}_{\downarrow\uparrow} \times M$ such that

$$\Omega_\downarrow - \Omega_\uparrow = \frac{F_\lambda}{2\pi i} \quad \text{on} \quad (\mathbb{T}_\downarrow \cap \mathbb{T}_\uparrow)^{(1)} \times M.$$

  The form defined locally by $d\Omega$ is now a global 3-form over $\mathbb{T} \times M$.
  Denote it by $H \leftarrow$ this form is cohomologous to $\frac{d\phi}{2\pi} \wedge \frac{F_\lambda}{2\pi i}$.

Realization of $\mathbf{F}$ as local Fock Bundles

- A Fock space $\mathcal{F}$ is an infinite dimensinal complex Hilbert space. It has a vacuum vector

$$|0\rangle = u_{-1} \wedge u_{-2} \wedge u_{-3} \wedge \cdots$$

and a base can be chosen by

$$u_{a_1} \wedge \cdots \wedge u_{a_k} \wedge |0\rangle_{b_1,\ldots,b_l}, \qquad a_1 > \ldots > a_k \geqslant 0$$
$$|0\rangle_{b_1,\ldots,b_l} = \text{vacuum with states } 0 > b_1 > \ldots > b_l \text{ removed.}$$

This is a charge $k - l$ vector.

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N: computes a charge of a state, $N = k$ in $\mathcal{F}^{(k)}$.

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- The loop algebra $\mathfrak{lt}$ (Lie algebra of $L\mathbb{T}$) has a projective irreducible highest weight representation on $\mathcal{F}$:

$$[e_n, e_m] = n\delta_{n,-m}.$$

- Take $\mathbf{F}_{\downarrow\uparrow}$ to be bundles of Fock spaces over $\mathbb{T}_{\downarrow\uparrow} \times M$ such that its charge k subbundle transforms as the linebundle $\lambda^{\otimes k}$.

Note. This means that if h_{ab} are some transition functions of $\lambda \to M$ (wrt the cover $\{U_a\}$), then the transition functions of $\mathbf{F}_{\downarrow\uparrow}$ are $(h_{ab})^N$.

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- Fix $\mathbb{T}$-valued local sections $s_a : U_a \to \lambda$.
  Define a family of unitary operators

$$u_a : (\mathbb{T}_\downarrow \cap \mathbb{T}_\uparrow)^{(1)} \times U_a \to U(\mathcal{F})$$
$$u_a(x) = s_a(x) \cdot S$$

- S creates a state of topological type λ over M
- then, topologically, we get the isomorphisms

$$\mathbf{F}_\uparrow \simeq \mathbf{F}_\downarrow \otimes \lambda \quad \text{over} \quad (\mathbb{T}_\downarrow \cap \mathbb{T}_\uparrow)^{(1)} \times M$$

Supercharges on $\mathbb{T} \times M$

- The supercharge is an unbounded and self-adjoint Fredholm operator defined by

$$Q_{\downarrow\uparrow}(x) = \sum_k \psi_k \otimes e_{-k} + \psi_0 \otimes \frac{\phi}{2\pi} \quad \text{on} \quad \mathbb{T}_{\downarrow\uparrow} \times M$$

- ψ_i are generators of the Clifford algebra $\mathrm{cl}(\mathfrak{lt})$ subject to

$$\{\psi_n, \psi_m\} = 2\delta_{n,-m}.$$

- the gerbe is tensored by a trivial Clifford-module bundle:

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- the Dixmier-Douady class is still the cup product.

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- The local families glue under conjugation by S_a:

$$S_a Q_{\downarrow}(2\pi, p) S_a^{-1} = Q_{\uparrow}(0, p)$$

  - this follows from the rules

$$S_a e_0 S_a^{-1} = (e_0 - 1) \qquad S_a e_n S_a^{-1} = e_n \qquad n \neq 0.$$

- In addition, pick any complex line bundle of finite rank ξ on M and take a tensor product with the gerbe:

$$\{\mathbf{S} \otimes \mathbf{F}_{\downarrow\uparrow} \otimes \xi\}$$

  and let Q act on it as $Q \otimes \mathbf{1}$.

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The supercharge has a representative in the twisted K-theory group $K^1(\mathbb{T} \times M, \tau)$:

$$\frac{Q_{\downarrow\uparrow}}{\sqrt{1 + (Q_{\downarrow\uparrow})^2}}.$$

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- - whats the role of ξ?
  - how does this correspond to

$$\{x \in K^1(M) : x \otimes \lambda = x\} \bigoplus_{\mu} \frac{K^0(M)}{(1-\lambda) \otimes K^0(M)}$$

Index of Q in Twisted Cohomology

- A twisted odd supercurvature is a pair of locally defined odd supercurvatures $\mathbb{F}_{\downarrow\uparrow}$ on $\mathbb{T}_{\downarrow\uparrow} \times M$

$$\mathbb{F}_{\downarrow\uparrow} = \mathbb{A}^2_{\downarrow\uparrow} + \Omega_{\downarrow\uparrow}.$$

  - $\mathbb{A}_{\downarrow\uparrow}$ are usual superconnections
  - recall that $\Omega_{\downarrow} - \Omega_{\uparrow} = F_{\lambda}$.

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  - $\mathbb{A}_{\downarrow\uparrow}$ are usual superconnections
  - recall that $\Omega_{\downarrow} - \Omega_{\uparrow} = F_{\lambda}$.
- We shall use a one parameter family of such superconnections

$$\mathbb{F}_{\downarrow\uparrow} = (\sqrt{t}\chi Q_{\downarrow\uparrow} + \nabla_{\downarrow\uparrow})^2 + \Omega_{\downarrow\uparrow}.$$

  - χ is a formal symbol with $\chi^2 = 1$ and it commutes with everything,
  - $t > 0$ real,
  - $\nabla_{\downarrow\uparrow}$ are the connections on $\mathbf{F}_{\downarrow\uparrow}$:

$$\nabla_{\downarrow\uparrow} = \nabla_{\xi} + N\nabla_{\lambda}.$$

- The odd twisted super-Chern character is the form

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The Chern character form $\textbf{ch-ind}_1^{\tau}$ is an odd differential form on $\mathbb{T} \times M$ and a cocycle in the twisted cohomology. The class in twisted cohomology is independent on the choice of a superconnection.

Twisted cohomology is computed from the usual de-Rham complex with the differential $d - H$.

- The odd twisted super-Chern character is the form

$$\mathbf{ch\text{-}ind}_1^\tau = \mathrm{sTr}(e^{-\mathbb{F}_{\downarrow\uparrow}})$$

-sTr applies the trace to the linear term in χ.

The Chern character form $\mathbf{ch\text{-}ind}_1^\tau$ is an odd differential form on $\mathbb{T} \times M$ and a cocycle in the twisted cohomology. The class in twisted cohomology is independent on the choice of a superconnection.

Twisted cohomology is computed from the usual de-Rham complex with the differential $d - H$.

- The independence on the superconnection means that if we are given another superconnection $\mathbb{A}'$, and a different connection on the twiting line bundle such that the associated 3-curvature is H', then

$$[\mathbf{ch}_\bullet^\tau(\mathbb{A})] \mapsto [\mathbf{ch}_\bullet^\tau(\mathbb{A}')]$$

under the canonical isomorphism of cohomology groups which sends the cohomology associated with $(\Lambda^\bullet(M), d - H)$ to the cohomology associated with $(\Lambda^\bullet(M), d - H')$. More precisely, if $H' = H + d\eta$, then the isomorphism is determined by $\xi \mapsto e^{-\eta} \wedge \xi$.

The ∞-time limit of the odd character over $\mathbb{T} \times M$ is the distribution valued odd differential form

$$\lim_{t \to \infty} \mathrm{sTr}(e^{-\mathbb{F}_{\downarrow\uparrow}}) = \sqrt{\pi}\delta(e_0 + \frac{\phi}{2\pi})\frac{d\phi}{2\pi} \wedge \mathrm{tr}_{\xi}(e^{-F_{\downarrow\uparrow} + \Omega_{\downarrow\uparrow}}).$$

The symbol $\delta(e_0 + \frac{\phi}{2\pi})$ denotes the Dirac delta distribution. The support of the characters localizes over the zero subspaces of the supercharge families.

Index of Q - Harju-Mickelsson 12

We adopt the following strategy

- (1) Choose a covering map

$$\pi : \mathbb{R} \times M \to \mathbb{T} \times M$$

The gerbe trivializes on $\mathbb{R} \times M$, then pull the supercharge

$$\pi^*(Q) : \mathbb{R} \times M \to \textbf{Fred}^{(1)}.$$

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- (2) Choose a superconnection on the cover

$$\mathbb{F} = (\sqrt{t}\chi\pi^*(Q) + \pi^*(\nabla))^2$$

and define the index-character in the usual way

$$\mathbf{ch\text{-}ind}_1 \circ \pi^*(\mathbb{F}) = \mathrm{sTr}(e^{-\mathbb{F}}).$$

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$$\textbf{ch-ind}_1 \circ \pi^*(\mathbb{F}) = \mathrm{sTr}(e^{-\mathbb{F}}).$$

- (3) Oberve that the dependence on the homotopy is of the form

$$\frac{d}{dt}\textbf{ch-ind}_1 \circ \pi^*(\mathbb{F}_t) = -d(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{F}_t})$$

- (4) If $x \in \mathbb{R}$ is the coordinate in the cover, observe that

$$e^{-\mathbb{F}_t(x+2\pi,p)} = e^{-\mathbb{F}_t(x,p)-F_\lambda} = e^{-\mathbb{F}_t(x,p)} \wedge \mathrm{ch}(\lambda).$$

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- (6) Choose a section $\psi : \mathbb{T} \times M \to \mathbb{R} \times M$ and pull the character to

$$\frac{H^{\mathrm{odd}}(\mathbb{T} \times M, \mathbb{Q})}{(1 - \mathrm{ch}(\lambda)) \wedge \frac{d\phi}{2\pi} \wedge \mathrm{ch}(K^0(M))}$$

  - compare with

$$\{x \in K^1(M) : x \otimes \lambda = x\} \bigoplus_\mu \frac{K^0(M)}{(1-\lambda) \otimes K^0(M)}$$

Associated with the supercharge Q with vacuum twist ξ there is a character map

$$\hat{\textbf{ch-ind}}^{\tau}_{1}(Q) \in \frac{H^{\mathrm{odd}}(\mathbb{T} \times M, \mathbb{Q})}{(1 - \mathrm{ch}(\lambda)) \wedge \frac{d\phi}{2\pi} \wedge \mathrm{ch}(K^0(M))}$$

whose equivalence class

$$\sqrt{\pi}\frac{d\phi}{2\pi} \wedge \mathrm{tr}_{\xi}(e^{-F_{\xi}}).$$

is invariant under smooth homotopies.

Example $\mathbb{T}^3$.

- Consider a vector bundle $\xi \to \mathbb{T}^2$ with Chern character equal to

$$\mathrm{ch}(\xi) = n - j d\theta_1 \wedge d\theta_2.$$

Fix λ such that

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- The character of Q with vacuum twist ξ:

$$\frac{d\phi}{2\pi} \wedge (n - j\frac{d\theta_1 \wedge d\theta_2}{2\pi i}) \quad \mathrm{mod} \quad k\frac{d\phi}{2\pi} \wedge \frac{d\theta_1 \wedge d\theta_2}{2\pi i}$$

- the twisted K-theory class of Q depends on the parameter j up to multiplets of k, this comes from the subgroup $\mathbb{Z}_k$.

Application of This?

- T-dual transformation in the case of a product manifold gives an isomorphism of groups

$$t : K^1(\mathbb{T} \times M, \alpha \smile \beta) \to K^0(P)$$

such that P is the circle bundle whose Euler class is determined by β.

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- Suppose that $K^1(\mathbb{T} \times M, \alpha \smile \beta)$ has a nontrivial torsion component.
- If $\eth$ is a Dirac operator on P twisted by some complex vector bundle ξ, then one can factor the index computation through the twisted K-theory:

$$\mathbf{ch}\text{-}\hat{\mathbf{ind}}_1^{\tau}(t^{-1} \circ \eth).$$

  This would reveal the torsion parts in $K^0(P)$.

Suspended Superchare

- The Atiyah-Singer suspension is a homotopy equivalence

$$\mathrm{susp} : \mathbf{Fred}^{(1)} \to \Omega\mathbf{Fred}^{(0)}$$

defined by

$$\begin{aligned} \mathrm{susp}(A) &= \cos(s) + iQ\sin(s) \qquad s \in [0,\pi] \\ &= \cos(s) + i\sin(s) \qquad s \in [\pi, 2\pi]. \end{aligned}$$

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- When applied fiberwise in a bundle of Fredholm operators, one gets a homomorphism

$$\alpha : K^1(\mathbb{T} \times M, \tau) \to K^0(\mathbb{T}^2 \times M, \tau).$$

Goal. Define characteristic classes for the suspended supercharges.

- The index can be computed using the superconnection tecnhiques. For this reson we need to make $\alpha \circ Q$ an off diagonal operator acting on $\mathbb{Z}_2$-grade spinors
  Consider the complexified Clifford algebra subject to

$$\{\psi^i, \psi^j\} = -2\delta^{ij} \qquad i, j \in \{0, 1\}$$

  Realize this as

$$\psi^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \psi^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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- Define a section of self-adjoint Fredholm operators

$$\begin{aligned} D^{\downarrow\uparrow} &= \cos(s)\psi^0 + \sin(s)Q\psi^1 \\ &= \begin{pmatrix} 0 & \mathrm{susp}(Q^{\downarrow\uparrow})^* \\ \mathrm{susp}(Q^{\downarrow\uparrow}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_-^{\downarrow\uparrow} \\ D_+^{\downarrow\uparrow} & 0 \end{pmatrix} \end{aligned}$$

Note. The families $D^{\downarrow\uparrow}$ are not θ-summable, i.e.

$$e^{-t(D^{\downarrow\uparrow})^2} \qquad \text{is not a trace class operator.}$$

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$$e^{-t(D^{\downarrow\uparrow})^2} \qquad \text{is not a trace class operator.}$$

- In fact

$$\begin{aligned}(D^{\downarrow\uparrow})^2 &= \cos^2(s) + \sin^2(s) Q^2 \qquad \text{if } s \in [0, \pi] \\ &= \mathbf{1} \qquad \text{if } s \in [\pi, 2\pi].\end{aligned}$$

Therefore, $(D^{\downarrow\uparrow})^2$ are invertible outside the submanifold

$$(-\epsilon, \pi + \epsilon) \times \mathbb{T} \times M \qquad (\epsilon \text{ small}).$$

I will study the twisted index problem over this submanifold
- the index character will localize here as a bump-form and can be extended to a bump-form over $\mathbb{T}^2 \times M$.

- Lift D to the covering space $\pi : \mathbb{T}_s \times \mathbb{R} \times M \to \mathbb{T}^2 \times M$:

$$\pi^*(D) : \mathbb{T}_s \times \mathbb{R} \times M \to \underline{\mathbf{Fred}}^{(0)}.$$

Define a one parameter family of superconnections for the lifted family

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Define a one parameter family of superconnections for the lifted family

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- The character form of the index bundle over $\mathbb{T} \times \mathbb{R} \times M$ is equal to

$$\hat{\mathbf{ch\text{-}ind}}^{\tau} = \mathrm{sTr}(e^{-\mathbb{A}^2})$$

- the even supertrace is applied here, which has the $t \to \infty$ limit

$$\lim_{t\to\infty} \mathrm{sTr}(e^{-\mathbb{A}^2}) = c\delta(e_0 + \frac{\phi}{2\pi})\delta(s - \frac{\pi}{2})ds \wedge \frac{d\phi}{2\pi} \wedge \mathrm{tr}_\xi(e^{-\pi^*(F)})$$

- c is a constant which will be fixed later.

Character of the Index

- The pullback of the character form to $\mathbb{T}^2 \times M$ becomes well defined if we quotient the rational cohomology by the normal subgroup

$$(1 - \mathrm{ch}(\lambda)) \wedge \frac{d\phi}{2\pi} \wedge \mathrm{ch}(K^1(\mathbb{T}_s \times M)).$$

recall that $K^0(\mathbb{T}^2 \times M, \tau)$ is isomorhic to

$$\{x \in K^0(\mathbb{T}_s \times M) : x \otimes \lambda = x\} \bigoplus_{\mu} \frac{K^1(\mathbb{T}_s \times M)}{(1 - \lambda) \otimes K^1(\mathbb{T}_s \times M)}$$

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The character of the index of a suspended supercharge is equal to

$$\begin{aligned} \textbf{ch-ind}(\mathrm{susp}(Q)) &= cds \wedge \frac{d\phi}{2\pi} \wedge \mathrm{tr}_\xi(e^{-F_\xi}) \\ &\in \frac{H^{\mathrm{even}}(\mathbb{T}^2 \times M, \mathbb{Q})}{(1 - \mathrm{ch}(\lambda)) \wedge \frac{d\phi}{2\pi} \wedge \mathrm{ch}(K^1(\mathbb{T}_s \times M))}. \end{aligned}$$

- Define the de-suspension map Σ^{-1}

$$\Sigma^{-1}(\Omega) = \frac{i}{2\pi}\int_{\mathbb{T}_s} \Omega : \quad \frac{H^{\text{even}}(\mathbb{T}^2 \times M, \mathbb{Q})}{(1-\mathrm{ch}(\lambda)) \wedge \frac{d\phi}{2\pi} \wedge \mathrm{ch}(K^1(\mathbb{T}_s \times M))}$$
$$\to \quad \frac{H^{\text{odd}}(\mathbb{T} \times M, \mathbb{Q})}{(1-\mathrm{ch}(\lambda)) \wedge \frac{d\phi}{2\pi} \wedge \mathrm{ch}(K^0(M))}.$$

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$$\to \frac{H^{\text{odd}}(\mathbb{T} \times M, \mathbb{Q})}{(1-\mathrm{ch}(\lambda)) \wedge \frac{d\phi}{2\pi} \wedge \mathrm{ch}(K^0(M))}.$$

The odd character is the map

$$\mathbf{ch\text{-}\hat{ind}}_\tau^1 = c\Sigma^{-1} \circ \mathbf{ch\text{-}\hat{ind}}_\tau \circ \mathrm{susp}(Q)$$

Supercharge Suspension

Problem.
Given an odd Dirac family $\eth$ on X, there is a Dirac suspension, i.e. a Dirac family $\eth_s$ on $\mathbb{T} \times X$ which is represented in K^0 by the same element as $\mathrm{susp} \circ \eth$.
- what is the analogue of this in the twisted K-theory?
- this is an application of the representation theory of $\mathfrak{lt}^2$.

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- Consider a new copy of the unit circle $\mathbb{T}_s$ and define a Fock bundle

$$\mathbf{F}_s \to \mathbb{T}_s$$

- make the charge grows by 1 under translations around the cicle.
- this leads to a spectral flow around $\mathbb{T}_s$.

Define the local tensor product bundles

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This is a gerbe with Dixmier-Douady class $\tau = \alpha \smile \beta$.

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  This is a gerbe with Dixmier-Douady class $\tau = \alpha \smile \beta$.
- We have two copies of projective representations for $\mathfrak{lt}$
  - this gives a projective representation of $\mathfrak{lt}^2$:

$$[e_n, f_m] = 0, \qquad [e_n, e_m] = [f_n, f_m] = n\delta_{n,-m}.$$

- Consider the real Clifford algebra $\mathrm{cl}(\mathfrak{t}^2)$ which is the polynomial algebra generated by ψ_n^i with $i = 0, 1$ and $n \in \mathbb{Z}$ subject to the relations

$$\{\psi_n^i, \psi_m^j\} = 2\delta^{ij}\delta_{n,-m}.$$

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- Define a vacuum representation for cl$(\mathfrak{lt}^2)$,
  - there is a two dimensional vacuum where $\{\psi_0^0, \psi_0^1\}$ restict to cl(2).
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- Tensor the gerbe with a trivial bundle of $\mathrm{cl}(\mathfrak{lt}^2)$-modules. Tensor with an arbitrary rank complex vector bundle

$$\mathbf{S} \otimes (\mathbf{F}_s \boxtimes \mathbf{F}^{\downarrow\uparrow}) \otimes \xi,$$

  - this bundle is $\mathbb{Z}_2$-graded (because **S** is).

- The even supercharge is the Fredholm section

$$Q_s^{\downarrow\uparrow} : \mathbb{T} \times \mathbb{T}_{\downarrow\uparrow} \times M \to \mathbf{Fred}^{(0)}$$

defined by

$$Q_s^{\downarrow\uparrow}(s, \phi, p) = \sum_k \psi_k^0 \otimes e_{-k} + \sum_k \psi_k^1 \otimes f_{-k} + \psi_0^0 \phi \otimes \mathbf{1} + \psi_0^1 s \otimes \mathbf{1}.$$

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The section $Q_s^{\downarrow\uparrow}$ determines an element in the twisted K-theory group $K^0(\mathbb{T}^2 \times M, \tau)$ and its index character is

$$\mathbf{ch\text{-}\hat{ind}}^{\tau} = \frac{ds}{2\pi} \wedge \frac{d\phi}{2\pi} \wedge \mathrm{ch}(\xi).$$

Twisted K-Theory on $\mathbb{T}^3$

- Recall that $K^\bullet(\mathbb{T}_\phi \times \mathbb{T}^2, \tau)$ is isomorphic to

$$\{x \in K^\bullet(\mathbb{T}^2) : x \otimes \lambda = x\} \oplus \frac{K^{\bullet+1}(\mathbb{T}^2)}{(1-\lambda) \otimes K^{\bullet+1}(\mathbb{T}^2)}$$

All the twisted K^0 and K^1 classes associated to the second summand are already known,
- remains to study the invariant part.

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  All the twisted K^0 and K^1 classes associated to the second summand are already known,
  - remains to study the invariant part.
- No torsion in $K^\bullet(\mathbb{T}^3) \Rightarrow$ all the K-theory classes can be represented by a differential form.
  Fix the angle coordinates θ_1 and θ_2 for the circles in $\mathbb{T}^2$.
  Fix a twisting line bundle λ with a curvature equal to

$$\frac{1}{2\pi i} d\theta_1 \wedge d\theta_2.$$

  The invariant 1-forms satisfying

$$x \wedge \mathrm{ch}(\lambda) = x \qquad \text{in cohomology}$$

  are $x = d\theta_1$ and $x = d\theta_2$.

Torus - invariant part

- (1) Start with the local Fock bundles $\mathbf{F}_{\downarrow\uparrow} \to \mathbb{T}_{\downarrow\uparrow} \times \mathbb{T}^2$
  - defines a gerbe associated with the DD-class τ.

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- (2) Construct another Fock bundle $\mathbf{F}_1 \to \mathbb{T}_\phi \times \mathbb{T}^2$ such that
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- (5) The zeros are on the submanifold $\mathbb{T}_\phi \times \{\theta_1 = 0\} \times \mathbb{T}_{\theta_2}$:

$$\begin{aligned}(\tilde{Q}_{\downarrow\uparrow})^2 &= \sum_{k>0} \Big[k\psi_{2k-1}\psi_{-2k+1} + 2e_k e_{-k} + \\ &+ k\psi_{2k}\psi_{-2k} + 2f_k f_{-k} + (f_0 + \theta_1)^2 \Big].\end{aligned}$$

The supercharge operators $\widetilde{Q}_{\downarrow\uparrow}$ determine a class in a K-theory of $\mathbb{T}^{n+1}$ twisted by the gerbe

$$\mathbf{S} \otimes \mathbf{F}_{\downarrow\uparrow} \otimes \mathbf{F}_1 \otimes \xi$$

- the conjugation action by S_ϕ on $\widetilde{Q}_{\downarrow\uparrow}$ is the identity action since

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$$S_\phi e_n S_\phi^{-1} = e_n \text{ if } n \neq 0.$$

The characteristic class gives

$$\textbf{ch-}\hat{\textbf{ind}}^\tau(Q) = \frac{d\theta_1}{2\pi}$$

- this form would be well defined even in $H^{\text{odd}}(\mathbb{T}^{n+1})$,
- to see this, pick a superconnection $\mathbb{A}$, and then

$$S_\phi \text{sTr}(e^{-\mathbb{A}^2}) S_\phi^{-1} = \text{sTr}(e^{-\mathbb{A}^2}) \wedge e^{-F_\lambda}$$

but, since $d\theta_1 \wedge \mathrm{ch}(\lambda) = d\theta_1$, we have

$$S_\phi \text{sTr}(e^{-\mathbb{A}^2}) S_\phi^{-1} = \text{sTr}(e^{-\mathbb{A}^2}).$$

Note.

- for higher dimensional tori, tensor the Hilbert bundles with complex vector bundles

$\Rightarrow$ a complete geometric realization of $K^1(\mathbb{T}^{n+1}, \tau)$.

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- even theory, $K^0(\mathbb{T}^{n+1}, \tau)$, can be solved with similar methods.