DIRAC OPERATORS ON 4-MANIFOLDS

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1. The Dirac Operator on $\mathbb{R}^4$ and Clifford Algebra

Dirac operators are important geometric operators on a manifold. The Dirac operator D_A on the four dimensional Euclidean space $M = \mathbb{R}^4$ is the order one differential operator whose square $D_A \circ D_A$ is the Euclidean Laplacian $-\sum_{i=1}^4 \frac{\partial^2\psi}{\partial x_i^2}$. However, this is not possible unless we allow coefficients for this linear operator to be matrix-valued. Let $M = \mathbb{R}^4$ be the four dimensional Euclidean space with global Euclidean coordinate (x_1, x_2, x_3, x_4) and $W \otimes L = M \times \mathbb{C}^4$ be the trivial 4-dimensional complex vector bundle over M, then the Dirac operator $D_A : C^\infty(\mathbb{R}^4, \mathbb{C}^4) \to C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ is given by

$$D_A\psi = \sum_{i=1}^4 e_i \cdot \frac{\partial\psi}{\partial x_i} \tag{1.1}$$

where $e_i \in \mathrm{End}(\mathbb{C}^4)$ is a 4×4 complex matrix satisfying $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}$. Explicitly, we take

$$e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

and

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Let V be the dimension 4 Euclidean space of complex linear homomorphism from W_+ to W_- with orthonormal basis

$$\mathbf{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathbf{e_4} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Denote by W_+, W_- two copies of $\mathbb{C}^2$ with the standard hermitian metric. Let $W = W_+ \oplus W_-$. Regard elements of V as complex linear homomorphisms from W_+ to W_- by sending $\mathbf{e_1}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ etc. Construct the complex linear map

$$\theta : V \otimes \mathbb{C} \to \mathrm{End}(W) \qquad \theta(Q) = \begin{pmatrix} 0 & -\bar{Q}^T \\ Q & 0 \end{pmatrix} \qquad \theta(\mathbf{e_i}) = e_i. \tag{1.2}$$

Definition 1.1. *The (complex) Clifford algebra of the dimension 4 Euclidean space V is $\mathbb{C}l(V) = \mathrm{End}(W)$ the 16-dimensional algebra over $\mathbb{C}$ generated by e_1, e_2, e_3, e_4 subject to the relations $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}$.*

Remark 1.2. As a vector space, $\mathbb{C}l(V) = \mathrm{End}(W) = \Lambda^0 V \otimes \mathbb{C} \oplus \cdots \oplus \Lambda^4 V \otimes \mathbb{C}$. The Clifford multiplication of e_i on forms is given by

$$e_i \cdot w = e_i \wedge w - \iota(e_i)w \qquad \iota(e_i) : \Lambda^k V \to \Lambda^{k-1} V \qquad (\iota(e_i)w, \theta) = (w, e_i \wedge \theta). \tag{1.3}$$

2. *Spin*^{c} Group and *spin*^{c}-structure

Let G be a Lie group and $r : G \to GL(\mathbb{F}^n)$ be a representation on a vector space $\mathbb{F}^n$.

Definition 2.1. *A vector bundle E over a manifold M has a G-*structure *if there is an open covering $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of M and for each $\alpha, \beta \in I$, there is a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying the cocycle condition*

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}, \qquad \text{on} \qquad U_\alpha \cap U_\beta \cap U_\gamma, \tag{2.1}$$

*so that E is isomorphic to $\sqcup_{\alpha \in I} U_\alpha \times \mathbb{F}^n / \sim$ where $(u_\alpha, v_\alpha) \sim (u_\beta, v_\beta)$ if $u_\alpha = u_\beta \in U_\alpha \cap U_\beta$ and $v_\alpha = g_{\alpha\beta} v_\beta$. A manifold M has a G-*structure *if the tangent bundle TM has a G-structure.*

Example 2.2. Any real vector bundle of dimension m has a $GL(\mathbb{R}^m)$-structure.

Example 2.3. A complex line bundle corresponds a $U(1)$-structure: $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$ with the cocycle condition.

Example 2.4. If M is an oriented n-dimensional Riemannian manifold, then the tangent bundle TM of M carries a $SO(n)$-structure, i.e., there exists an open cover $\{U_\alpha : \alpha \in I\}$ of M and a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)$ for each $\alpha, \beta \in I$ satisfying the cocycle condition so that TM is isomorphic to the gluing of the trivial $\mathbb{R}^n$ bundles over U_α by the gluing function $g_{\alpha\beta}$.

We shall now introduce the spin group $Spin(4)$ and the *spin*^{c}-group $Spin(4)^c$ which will be used to define *spin*^{c} structures on a four dimensional manifold M.

Let $V = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & z \end{pmatrix} : z, w \in \mathbb{C} \right\}$ be the Euclidean 4-space with the natural inner product $\langle X, Y \rangle = \bar{X}^T Y$. The *special unitary group* $SU(2)$ is defined as

$$SU(2) = \left\{ Q = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in V : \det{}^{\frac{1}{2}} [\bar{Q}^T Q] = |z|^2 + |w|^2 = 1 \right\}. \tag{2.2}$$

Definition 2.5. *The $Spin(4)$ group is the dimension 6 Lie group $SU(2) \times SU(2)$*

$$Spin(4) = \left\{ \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} : A_\pm \in SU(2) \right\}. \tag{2.3}$$

The $Spin(4)^c$ group is the dimension 7 Lie group

$$Spin(4)^c = \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} : A_\pm \in SU(2), \lambda \in U(1) \right\}. \tag{2.4}$$

$Spin(4)$ naturally represents on V via $(A_+, A_-) \mapsto (Q \mapsto A_- Q A_+)$ and it is straight forward to check that the representation preserves the inner product on V, i.e.,

$$\langle A_- Q A_+, A_- Q A_+ \rangle = \langle Q, Q \rangle. \tag{2.5}$$

Hence the representation gives rise to a surjective homomorphism

$$\rho : Spin(4) \to SO(V) \cong SO(4). \tag{2.6}$$

The kernel of the homomorphism is $\{(I, I), (-I, -I)\}$. Then we have the short exactly sequence

$$0 \to \mathbb{Z}_2 \to Spin(4) \to SO(4) \to 0. \tag{2.7}$$

The group $Spin(4)$ is a double cover of $SO(4)$. Topologically, $Spin(4)$ is the simply connected manifold $S^3 \times S^3$.

Similarly, the group $Spin(4)^c$ has the representation

$$\rho^c : Spin(4)^c \to SO(V) \cong SO(4) \qquad \rho^c(\begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix})(Q) = (\lambda A_-) Q (\lambda A_+)^{-1} \tag{2.8}$$

In addition, there is a group homomorphism

$$\pi : Spin(4)^c \to U(1) \qquad \pi(\begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix}) = \det(\lambda A_+) \det(\lambda A_-) = \lambda^2. \tag{2.9}$$

Then, we obtain the following short exact sequence

$$0 \to \mathbb{Z}_2 \to Spin(4)^c \to SO(4) \times U(1) \to 0. \tag{2.10}$$

The group $Spin(4)^c$ admits two irreducible unitary representations ρ_+, ρ_- on the 2-dimensional complex vector space W_+, W_- by

$$\rho \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (w_\pm) = \lambda A_\pm w_\pm \qquad w_\pm \in W_\pm. \tag{2.11}$$

As unitary-length element of V are 2×2 unitary matrices, they can be regarded isometries from W_+ to W_-, we have an isomorphism of representation spaces

$$V \otimes \mathbb{C} \cong \mathrm{Hom}(W_+, W_-). \tag{2.12}$$

Definition 2.6. *Let M be a 4-dimensional Riemannian manifold and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(4)$ is the associated $SO(4)$-structure of the tangent bundle TM. A $Spin^{(c)}$ structure on M is given by an open covering $\{U_\alpha : \alpha \in I\}$ of M and a collection of transition functions $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(4)^{(c)}$ such that $\rho^{(c)} \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition is satisfied.*

Example 2.7. If M has a spin structure defined by the transition function

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \mapsto Spin(4) \tag{2.13}$$

and L is a complex line bundle over M with hermitian metric and transition functions

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1) \tag{2.14}$$

Then a *spin*^{c}-structure on M can be obtained by taking the transition function

$$\tilde{g}_{\alpha\beta} h_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(4)^c. \tag{2.15}$$

Definition 2.8. *Let M be a 4-dimensional spin^c-manifold where the transition functions are given by $\tilde{g}_{\alpha\beta}$ for every $\alpha, \beta \in I$. Then the 2-dimensional complex vector bundles given by the transition functions $\rho \circ \tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to SU(2)$ are called the* spin bundles*, denoted $W_\pm \otimes L$. The complex line bundle given by the transition function $\pi \circ \tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$ is called the* determinant line bundle*, denoted L^2. $W_\pm$ and L are called virtual vector bundles.*

Let V be the 4-dimensional vector space as above, then the tangent bundle TM for a *spin*^{c} four manifold M is isomorphic to $\sqcup_\alpha U_\alpha \times V / \sim$ where $(u_\alpha, v_\alpha) \sim (u_\beta, v_\beta)$ if $u_\alpha = u_\beta$ and $v_\alpha = \tilde{g}_{\alpha\beta} v_\beta$. Therefore, in view of (2.12), we have

$$TM \otimes \mathbb{C} \cong Hom(W_+ \otimes L, W_- \otimes L). \tag{2.16}$$

3. Characteristic Classes and $Spin^c$-Structure

In this section[1], we introduce orientation, spin structure and *spin*^{c} structure of a 4-manifold from the point of view of algebraic topology. We shall first review Cech cohomology $H^*(M, \underline{G})$ of a manifold M with coefficient in a sheaf $\underline{G}$ of an abelian group G on M.

Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be a open cover of M. Associate a simplicial complex to the cover $\mathcal{U}$, i.e., $[\alpha_0 \alpha_2 \cdots \alpha_n]$ spans an n-simplex in $C_n(\mathcal{U})$ if and only if $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \emptyset$. Let G be an abelian group and $\underline{G}$ denotes the sheaf of G. Define the n-cochains to be the set $C^n(\mathcal{U})$ of continuous sections on $C_n(\mathcal{U})$ with coefficients in $\underline{G}$:

$$f_{\alpha_0 \cdots \alpha_n} : U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \to \underline{G} \tag{3.1}$$

which satisfy $f_{\cdots\alpha\cdots\beta\cdots} = f^{-1}_{\cdots\beta\cdots\alpha\cdots}$. The n-th coboundary map $\delta_n : C^n(\mathcal{U}) \to C^{n+1}(\mathcal{U})$ is given by

$$[\delta_n f]_{\alpha_0 \cdots \alpha_{n+1}} = \Pi f^{(-1)^i}_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_n}. \tag{3.2}$$

The n-th Cech cohomology of the cover $\mathcal{U}$ is defined by

$$H^n(\mathcal{U}, \underline{G}) = [ker \delta_n : C^n(\mathcal{U}) \to C^{n+1}(\mathcal{U})] / [im \delta_{n-1} : C^{n-1}(\mathcal{U}) \to C^n(\mathcal{U})]. \tag{3.3}$$

[1] I would like to thank my colleagues Nick Buchdahl and David Roberts for clarifying a few issues of the lecture notes.

The Cech cohomology of M with coefficient in $\underline{G}$ is defined by the direct limit of the group $H^n(\mathcal{U}, \underline{G})$ over finer and finer coverings. If we assume in addition that any finite intersection of elements of the open cover $\mathcal{U}$, the p-th cohomology of this intersection with values in $\underline{G}$ is 0 of all $p > 0$. Then the Cech cohomology of M and the Cech cohomology of the cover $\mathcal{U}$ coincide:

$$H^n(X, \underline{G}) \cong H^n(\mathcal{U}, \underline{G}) \tag{3.4}$$

An open cover $\mathcal{U}$ satisfying this condition is called a *good cover*. In the situations to be used in this talk, this condition can be replaced by $U_{\alpha_0} \cap U_{\alpha_2} \cap \cdots U_{\alpha_n}$ is either empty or contractible. The simplicial complex corresponding to a good cover reflects the topological feature of M. For example, when $M = S^1$, one can find a good cover whose corresponding simplicial complex is a triangle.

Remark 3.1. Let $\mathcal{U}$ be a good cover of M and let G be a discrete abelian group. The singular cohomology $H^n_{sing}(M, \underline{G})$ is isomorphic to the Cech cohomology $H^n(M, \underline{G})$.

To simplify the notation, in the following we shall now not distinguish the notation for the sheet $\underline{G}$ and the group G.

Example 3.2 (Principal G-bundle/Vector bundle with G-structure). f is a 1-cocycle if $f \in C^1(\mathcal{U})$ and $\delta_1 f = e$, i.e.,

$$f_{\alpha\beta} : U_\alpha \cap U_\beta \to G \qquad (\delta_1 f)_{\alpha\beta\gamma} = f_{\beta\gamma} f_{\alpha\gamma}^{-1} f_{\alpha\beta} = e. \tag{3.5}$$

If a vector bundle E over M has a G-structure, then E gives rise to a class

$$[E] \in H^1(X, G). \tag{3.6}$$

Note that if G is not abelian, $H^1(M, G)$ still makes sense, but it is just a set of equivalence classes of principal G-bundles over M (vector bundles with G-structure).

Example 3.3 (First Chern class of a complex line bundle). Let E be a complex line bundle over a manifold M. Then E has a $U(1)$-structure, in other words, $[E] \in H^1(M, U(1))$. Then we have a 1-cocycle $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$ with cocycle condition $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$. Then there exists $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}$ such that $g_{\alpha\beta} = e^{i2\pi h_{\alpha\beta}}$. Note that $h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} \in \mathbb{Z}$ and may not be 0. So h is not a 1-cocycle. However, $h_{\alpha\beta\gamma} = h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}$ is a 2-cocycle (check by exercise $(\delta_2 h)_{\alpha\beta\gamma\delta} = h_{\beta\gamma\delta} - h_{\alpha\gamma\delta} + h_{\alpha\beta\delta} - h_{\alpha\beta\gamma}$ is always equal to 0). The 2 cocycle $h_{\alpha\beta\gamma}$ gives rise to a class in $H^2(X, \mathbb{Z})$. This is referred to the first Chern class of E denoted by $c_1(E)$. We remark that the first Chern class map on complex line bundles over M gives rise to an isomorphism

$$H^1(M, U(1)) \cong H^2(M, \mathbb{Z}). \tag{3.7}$$

Example 3.4 (Second Stiefel-Whitney class). Let E be a vector bundle over M which carries a $SO(4)$-structure, then there exists 1-cocycle $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(4)$. As $Spin(4)$ is a 2-fold cover of $SO(4)$ and $U_\alpha \cap U_\beta$ is contractible, we can locally lift $g_{\alpha\beta}$ to $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(4)$. However, this is not a 1-cocycle as the cocycle condition $\tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} = \tilde{g}_{\alpha\gamma}$ may not be satisfied. However, we can construct a 2-cocycle by

$$\tilde{g}_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} \tilde{g}_{\gamma\alpha} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}_2 \subset Spin(4). \tag{3.8}$$

Check as an exercise that the cocycle condition

$$(\delta_2 \tilde{g})_{\alpha\beta\delta\gamma} = \tilde{g}_{\beta\gamma\delta} \tilde{g}^{-1}_{\alpha\gamma\delta} \tilde{g}_{\alpha\beta\delta} \tilde{g}^{-1}_{\alpha\beta\gamma} = e \tag{3.9}$$

is satisfied. The class of this 2-cocycle $\tilde{g}_{\alpha\beta\gamma}$ in $H^2(M, \mathbb{Z}_2)$ is called the second Stiefel-Whitney class of the vector bundle E, denoted $w_2(E)$.

Lemma 3.5. *Let $\{e\} \to K \to G \to G' \to \{e\}$ be a short exact sequence of groups where K is abelian. Then the following natural map is exact*

$$H^1(M, G) \to H^1(M, G') \to H^2(M, K). \tag{3.10}$$

Theorem 3.6. *If M is an oriented Riemannian 4-manifold, then M has a spin structure if and only if the second Stiefel-Whitney class $w_2(TX)$ vanishes.*

Proof. As $0 \to \mathbb{Z}_2 \to Spin(4) \to SO(4) \to 0$, then by Lemma 3.5 we have

$$H^1(M, Spin(4)) \to H^1(M, SO(4)) \to H^2(M, \mathbb{Z}_2) \tag{3.11}$$

exact. One observe that the vanishing of $w_2(TM)$ means that $[TM] \in H^1(M, SO(4))$ is an image of some element of $H^1(M, Spin(4))$. $\square$

Theorem 3.7. *If M is an oriented Riemanian 4 manifold, then M has a spin^c structure if and only if there exists a complex line bundle E over M, so that the $\mathbb{Z}_2$ reduction of its first Chern class is equal to the second Stiefel-Whitney class of TM.*

Proof. Denote by $r_* : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2)$ the $\mathbb{Z}_2$-reduction. Note that the reduction of the first Chern class is equal to the second second Stiefel-Whitney class because of the following commuting diagram:

$$\begin{array}{ccccc} H^1(M, U(1)) & \xrightarrow{\sigma} & H^1(M, U(1)) & \xrightarrow{w_2} & H^2(M, \mathbb{Z}_2) \\ {\scriptstyle c_1}\downarrow & & {\scriptstyle c_1}\downarrow & & {\scriptstyle =}\downarrow \\ H^2(M, 2\mathbb{Z}) & \xrightarrow{\times 2} & H^2(M, \mathbb{Z}) & \xrightarrow{r_*} & H^2(M, \mathbb{Z}_2) \end{array}$$

as a result of the short exact sequences $0 \to \mathbb{Z}_2 \to U(1) \to U(1) \to 0$ and $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. As $0 \to \mathbb{Z}_2 \to Spin(4)^c \to SO(4) \times U(1) \to 0$, we have the following exact sequence of homomorphisms:

$$H^1(M, Spin(4)) \to H^1(M, SO(4)) \oplus H^1(M, U(1)) \to H^2(M, \mathbb{Z}_2). \tag{3.12}$$

The manifold having a spin structure means that the existence of a spin structure for a principal $Spin(4) \times U(1)$-bundle, i.e., $w_2(TM) + w_2(E) = 0$. The theorem is proved when we observe that the vanishing of $w_2(TM)+r_*c_1(E)$ means that $[TM] \in H^1(M, SO(4))$ is an image of some element of $H^1(M, Spin(4)^c)$. $\square$

Remark 3.8. Spin^c structure is a topological property true for many manifolds. For example, an (almost) complex manifold M (dimension n) has a canonical spin^c structure. In fact, the line bundle $L = \Lambda^n TM$ over M has the same first Chern character as that of the tangent bundle TM. Moreover, $c_1(TM)$ is a canonical lift of $w_2(X)$ to $H^2(M, \mathbb{Z})$) (to see this, one needs to generalise $p_*(c_1(E)) = w_2(E)$ from a complex line bundle E to complex bundles of any dimension). Thus $p_*(c_1(\Lambda^n TM)) = w_2(TM)$ and by Theorem 3.7, the (almost) complex manifold M has a spin^c structure.

Theorem 3.9. *Every oriented four manifold admits a spin^c-structure.*

Proof. We shall prove the case when the manifold is simply connected. A proof in the general case can be found in *Spin Geometry* Appendix D by Lawson and Michelsohn. There is also a short proof by Teichner and Vogt.

The first integral homology for a simply connected manifold M vanishes, i.e., $H_1(M, \mathbb{Z}) \cong 0$. By the poincare duality $H^3(M, \mathbb{Z}) \cong H_1(M, \mathbb{Z})$, we have $H^3(M, \mathbb{Z}) \cong 0$. The short exact sequence $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ gives rise to the following long exact sequence

$$\cdots \to H^2(M, \mathbb{Z}) \xrightarrow{p_*} H^2(M, \mathbb{Z}_2) \to H^3(M, \mathbb{Z}) \to \cdots. \tag{3.13}$$

The vanishing of the third cohomology of M implies that the $\mathbb{Z}_2$ reduction map p_* is surjective. Thus, given the second Stiefel-Whitney class $w_2(TM)$ in $H^2(M, \mathbb{Z}_2)$ we know the existence of a pre-image $\eta \in H^2(M, \mathbb{Z})$. In view of (3.7), this implies the existence of a complex line bundle L whose first Chern class is $c_1(L) = \eta$. Hence the $\mathbb{Z}_2$-reduction of $c_1(L)$ coincides with $w_2(TM)$. Therefore, the manifold M carries a spin^c structure by Theorem 3.7. $\square$

4. Dirac Operators

Definition 4.1. *A connection on a vector bundle E (with G-strcuture) defined by a covering $\{U_\alpha : \alpha \in I\}$ and transition functions $\{g_{\alpha\beta} : \alpha, \beta \in I\}$ is a collection of differential operators*

$$\{d + \omega_\alpha : \alpha \in I\} \tag{4.1}$$

where d is the exterior derivative and ω_α takes values in the Lie algebra of G.

Let M be a 4-dimensional Riemannian manifold with a *spin*^{c} structure and a *spin*^{c} connection $d_A = \nabla^A$ on the spin bundle $W \otimes L$.

Definition 4.2. *The Dirac operator $D_A : \Gamma(W \otimes L) \to \Gamma(W \otimes L)$ with coefficient in the line bundle L is defined by*

$$D_A(\psi) = \sum_{i=1}^{4} e_i \cdot d_A\psi(e_i) = \sum_{e_i}^{4} e_i \cdot \nabla^A_{e_i}\psi. \tag{4.2}$$

where e_i is an orthonormal basis of TM.

Note that the Dirac operator divides into two pieces: $D_A^+ : \Gamma(W_+ \otimes L) \to \Gamma(W_- \otimes L)$ and $D_A^- : \Gamma(W_- \otimes L) \to \Gamma(W_+ \otimes L)$.

As in the Hodge theory, the theory of elliptic operators implies the kernels $kerD_A^+$ and $kerD_A^-$ are finite dimensional complex vector spaces. we define the index of D_A to be

$$\operatorname{ind} D_A^+ = \dim(\ker D_A^+) - \dim(\ker D_A^-). \tag{4.3}$$

Theorem 4.3 (Atiyah-Singer index theorem)**.** *If D_A is a Dirac operator with coefficients in a line bundle L on a compact oriented four manifold, then*

$$\operatorname{ind} D_A^+ = -\frac{1}{8}\tau(M) + \frac{1}{2}\int_M c_1(L)^2. \tag{4.4}$$

where $\tau(M) = b_+ - b_-$ is the signature of M, i.e., signature of the map

$$H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \to \mathbb{R} \qquad (x, y) \mapsto \int_M x \cup y. \tag{4.5}$$

Remark 4.4. If M is spin and W is a spin bundle, then for any complex line bundle L, the bundle $W \otimes L$ has a *spin*^{c}-structure. In this case, the determinant line bundle L^2 is indeed the tensor of L with L. However, if M is *spin*^{c} and not *spin*, the determinant line bundle L^2 make sense even though L is not a genuine bundle (in this case, L is called a virtual vector bundle). In this case we can replace $c_1(L)$ by $\frac{1}{2}c_1(L^2)$ using the following lemma. The proofs of the lemma simply follows from $g_{\alpha\beta} = e^{i2\pi h_{\alpha\beta}}$, the relation between the transition functions of a line bundle and the transition functions of its first Chern class.

Lemma 4.5. *Let L_1, L_2 be complex line bundle over a manifold M, then $L_1 \otimes L_2$ is also a complex line bundle and $c_1(L_1 \otimes L_2) = c_1(L_1) + c_2(L_2)$.*