Moduli of special Lagrangian and coassociative submanifolds

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This talk is based on

Moduli of coassociative submanifolds and semi-flat G_2-manifolds

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Contents

Mirror symmetry and SYZ

5 string theories and M-theory linked by dualities

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Strominger Yau Zaslow conjecture: X and Y are special Lagrangian fibrations over same base with dual fibres

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- coassociative $\iff$ coassociative ?
- coassociative $\iff$ special Lagrangian $+$ flux?

Contents

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Normed Algebras

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- $\mathbb{O}$ - octonions (8-d)

G_2 and the octonions

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Relation to the 3-form:

$$\phi(x, y, z) = \langle x \times y, z \rangle$$

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  (an associative subalgebra $\simeq \mathbb{H}$)
- $\mathcal{A}$ can be oriented such that $\phi|_{\mathcal{A}} = dvol_{\mathcal{A}}$
  i.e. $\mathcal{A}$ is a calibrated subspace with respect to the calibration ϕ

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equivalently: X is a 7-manifold with a G_2-structure such that

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- $\mathcal{C}$ is coassociative iff $\phi|_{\mathcal{C}} = 0$

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Theorem (McLean)

A normal vector field ν represents a first order deformation through coassociative submanifolds iff

$$\iota_\nu \phi|_{\mathcal{C}} \textit{ is closed,}$$

hence a harmonic self-dual form.

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notation: for $X \in T_{\mathcal{C}}\mathcal{M}$ let ω_X be the corresponding harmonic form

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$$\begin{aligned} g_{\mathcal{M}}(X, Y) &= \int_{\mathcal{C}} \omega_X \wedge \omega_Y \\ &= \langle [\omega_X] \smile [\omega_Y], [\mathcal{C}] \rangle \end{aligned}$$

called the L^2 moduli space metric

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Theorem

α is closed, so (locally) we have a function

$$u : \mathcal{M} \to H^2(\mathcal{C}, \mathbb{R})$$

such that $\alpha = du$:

$$u_*(X) = \alpha(X) = [\omega_X]$$

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$\mathcal{M}$ is immersed as a maximal positive definite submanifold of $H^2(\mathcal{C}, \mathbb{R})$

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then any coassociative fibration $\pi : X \to B$ must degenerate

(i.e. π can't be a submersion everywhere)

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Assume $F \to X \to B$ is a non-degenerate fibration

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Then B is a homotopy 3-sphere and F is simply connected

Proof (2)

Leray-Serre spectral sequence $\Rightarrow$ isomorphisms

$$i^* : H^2(X,\mathbb{R}) \to H^2(F,\mathbb{R})$$
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$$[\phi] = c\pi^*([dvol_B]), \;\; c \neq 0, \;\; \int_B dvol_B = 1$$

then for a closed 4-form μ on X

$$\int_X \mu \wedge \phi = c \int_F i^* \mu$$

Proof (3)

Recall: for a G_2-manifold (with $b^1(X) = 0$) the pairing $H^2(X, \mathbb{R}) \otimes H^2(X, \mathbb{R}) \to \mathbb{R}$ given by

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(Donaldson): intersection form on F is diagonal
(i.e. of form $\mathrm{diag}(-1, -1, \dots, -1)$)

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but this is

$$c \int_F i^* p_1(X) = c \int_F p_1(F) = 0$$

by Hirzebruch signature theorem $\Rightarrow$ contradiction

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one would hope that metric could be adjusted to produce genuine G_2-manifolds

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and $\pi : M \to \mathbb{CP}^2$ is a fibration by Cayley 4-folds

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ψ closed $\implies$ the fibres are minimal submanifolds, could we flow to a fibration of a G_2-manifold?

Riemannian submersion case

Theorem

Let $\pi : X \to B$ be a coassociative fibration with compact fibres

Riemannian submersion case

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in this case B is the moduli space of deformations and

$$g_B = \frac{1}{2\mathrm{vol}(F)} g_{L^2}$$

Proof

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follows that we can find Hyperkähler forms iff π is a Riemannian submersion

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π is a Riemannian submersion and B is the moduli space of deformations

(locally) we have the moduli space map

$$u : B \rightarrow H^2(T^4, \mathbb{R}) \simeq \mathbb{R}^{3,3}$$

Local form (1)

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- an oriented 3-manifold B
- $u : B \to H^2(T^4, \mathbb{R}) \simeq \mathbb{R}^{3,3}$ that maps TB to positive definite subspaces
- τ a positive constant (representing the fibre volume)

Local form (2)

Give B the metric

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define a 3-form ϕ on $B \times \mathbb{R}^4/\mathbb{Z}^4$

$$\phi = dvol_B + du$$

where $u : B \to H^2(T^4, \mathbb{R})$ is thought of as a 2-form

$$u(b) = u_{ij}(b) dx^{ij}$$

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$d\psi = 0$ iff u is a harmonic map
i.e. $u : B \to H^2(T^4, \mathbb{R})$ is a minimal immersion

all semi-flat fibrations have this form locally

Surface reduction

Suppose that $u : B \to \mathbb{R}^{3,3}$ is conical:

$$B = (0, \infty) \times \Sigma$$
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$$B = (0, \infty) \times \Sigma$$
$$g_B = dr^2 + r^2 g_\Sigma$$

this corresponds to a (positive definite) minimal surface in the quadric

$$Q = \{x \in \mathbb{R}^{3,3} | \langle x, x \rangle = 1\}$$

Affine Toda equations

A particular class of minimal surfaces into Q correspond to the equations:

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A particular class of minimal surfaces into Q correspond to the equations:

$$u_{z\bar{z}} = -e^{v-u} - e^{u},$$
$$v_{z\bar{z}} = q\bar{q}e^{-v} + e^{v-u}.$$

where q is a holomorphic cubic differential

- this is a set of affine Toda equations

THANK YOU