Moduli of special Lagrangian and coassociative submanifolds

David Baraglia

The Australian National University
Canberra, Australia

July 19, 2010
Moduli of coassociative submanifolds and semi-flat $G_2$-manifolds
arXiv:0902.2135v2
Contents

1 Some dualities

2 $G_2$ geometry

3 Deformations of coassociative submanifolds

4 Coassociative fibrations

5 Semi-flat coassociative fibrations
Mirror symmetry and SYZ

5 string theories and M-theory linked by dualities
5 string theories and M-theory linked by dualities

- type IIA - IIB duality leads to Mirror symmetry between Calabi-Yau manifolds $X \iff Y$
5 string theories and M-theory linked by dualities

- type IIA - IIB duality leads to Mirror symmetry between Calabi-Yau manifolds $X \iff Y$

Strominger Yau Zaslow conjecture: $X$ and $Y$ are special Lagrangian fibrations over same base with dual fibres
Gukov, Yau and Zaslow: the M-theory equivalent of a special Lagrangian fibration is a coassociative fibration of a $G_2$-manifold
Gukov, Yau and Zaslow: the M-theory equivalent of a special Lagrangian fibration is a coassociative fibration of a $G_2$-manifold by ‘brane reduction’:
Gukov, Yau and Zaslow: the M-theory equivalent of a special Lagrangian fibration is a coassociative fibration of a $G_2$-manifold by ‘brane reduction’:

- M-theory $K3$-fibration $\iff$ Heterotic $T^3$-fibration + flux
Gukov, Yau and Zaslow: the M-theory equivalent of a special Lagrangian fibration is a coassociative fibration of a $G_2$-manifold by ‘brane reduction’:

- M-theory $K3$-fibration $\iff$ Heterotic $T^3$-fibration + flux
- M-theory $T^4$-fibration $\iff$ IIB $T^3$-fibration + flux
More dualities

Gukov, Yau and Zaslow: the M-theory equivalent of a special Lagrangian fibration is a coassociative fibration of a $G_2$-manifold by ‘brane reduction’:

- M-theory $K3$-fibration $\iff$ Heterotic $T^3$-fibration + flux
- M-theory $T^4$-fibration $\iff$ IIB $T^3$-fibration + flux

so we would like to know about coassociative fibrations and the possibility of dual fibrations
Gukov, Yau and Zaslow: the M-theory equivalent of a special Lagrangian fibration is a coassociative fibration of a $G_2$-manifold by ‘brane reduction’:

- $M$-theory $K3$-fibration $\iff$ Heterotic $T^3$-fibration + flux
- $M$-theory $T^4$-fibration $\iff$ IIB $T^3$-fibration + flux

so we would like to know about coassociative fibrations and the possibility of dual fibrations

- coassociative $\iff$ coassociative?
- coassociative $\iff$ special Lagrangian + flux?
1. Some dualities

2. $G_2$ geometry

3. Deformations of coassociative submanifolds

4. Coassociative fibrations

5. Semi-flat coassociative fibrations
here $G_2$ means the compact form in $SO(7)$
A quick look at $G_2$

- here $G_2$ means the compact form in $SO(7)$
- stabiliser of a 3-form $\phi$ on $\mathbb{R}^7$
A quick look at $G_2$

- here $G_2$ means the compact form in $SO(7)$
- stabiliser of a 3-form $\phi$ on $\mathbb{R}^7$
- also preserves a 4-form $\psi = \ast \phi$
A quick look at $G_2$

- here $G_2$ means the compact form in $SO(7)$
- stabiliser of a 3-form $\phi$ on $\mathbb{R}^7$
- also preserves a 4-form $\psi = \ast \phi$
- standard 3-form:

$$\phi = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56})$$
A quick look at $G_2$

- here $G_2$ means the compact form in $SO(7)$
- stabiliser of a 3-form $\phi$ on $\mathbb{R}^7$
- also preserves a 4-form $\psi = \ast \phi$
- standard 3-form:
  \[ \phi = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56}) \]
- standard 4-form:
  \[ \psi = e^{4567} + e^{23} \wedge (e^{45} + e^{67}) + e^{31} \wedge (e^{46} - e^{57}) + e^{12} \wedge (-e^{47} - e^{56}) \]
An algebra over $\mathbb{R}$ is normed if it has an inner product such that

$$|xy| = |x||y|$$
Normed Algebras

An algebra over $\mathbb{R}$ is *normed* if it has an inner product such that

$$|xy| = |x||y|$$

there are only four:

- $\mathbb{R}$ - real numbers (1-d)
- $\mathbb{C}$ - complex numbers (2-d)
- $\mathbb{H}$ - quaternions (4-d)
- $\mathbb{O}$ - octonions (8-d)
An algebra over $\mathbb{R}$ is \textit{normed} if it has an inner product such that

$$|xy| = |x||y|$$

there are only four:

- $\mathbb{R}$ - real numbers (1-d)
Normed Algebras

An algebra over $\mathbb{R}$ is *normed* if it has an inner product such that

$$|xy| = |x||y|$$

there are only four:
- $\mathbb{R}$ - real numbers (1-d)
- $\mathbb{C}$ - complex numbers (2-d)
- $\mathbb{H}$ - quaternions (4-d)
- $\mathbb{O}$ - octonions (8-d)
An algebra over $\mathbb{R}$ is *normed* if it has an inner product such that

$$|xy| = |x||y|$$

there are only four:

- $\mathbb{R}$ - real numbers (1-d)
- $\mathbb{C}$ - complex numbers (2-d)
- $\mathbb{H}$ - quaternions (4-d)
An algebra over $\mathbb{R}$ is \textit{normed} if it has an inner product such that

$$|xy| = |x||y|$$

there are only four:

- $\mathbb{R}$ - real numbers (1-d)
- $\mathbb{C}$ - complex numbers (2-d)
- $\mathbb{H}$ - quaternions (4-d)
- $\mathbb{O}$ - octonions (8-d)
$G_2$ and the octonions

$G_2$ and $\mathbb{O}$ are closely related:

$G_2$ and $\mathbb{O}$ are closely related:
$G_2$ and the octonions

$G_2$ and $\mathcal{O}$ are closely related:

- $G_2 = \text{Aut}(\mathcal{O})$
$G_2$ and the octonions

$G_2$ and $\mathcal{O}$ are closely related:

- $G_2 = \text{Aut}(\mathcal{O})$
- $\text{Im}(\mathcal{O})$ is the 7-dimensional representation of $G_2$

Relation to the 3-form:

$$\phi(x, y, z) = \langle x \times y, z \rangle$$
$G_2$ and the octonions

$G_2$ and $\mathbb{O}$ are closely related:

- $G_2 = \text{Aut}(\mathbb{O})$
- $\text{Im}(\mathbb{O})$ is the 7-dimensional representation of $G_2$

Octonion multiplication gives rise to the cross product

$$\times : \text{Im}(\mathbb{O}) \otimes \text{Im}(\mathbb{O}) \to \text{Im}(\mathbb{O})$$

$$x \times y = \text{Im}(xy)$$
$G_2$ and $\mathbb{O}$ are closely related:

- $G_2 = \text{Aut}(\mathbb{O})$
- $\text{Im}(\mathbb{O})$ is the 7-dimensional representation of $G_2$

Octonion multiplication gives rise to the \textit{cross product}

$$\times : \text{Im}(\mathbb{O}) \otimes \text{Im}(\mathbb{O}) \to \text{Im}(\mathbb{O})$$

$$x \times y = \text{Im}(xy)$$

Relation to the 3-form:

$$\phi(x, y, z) = \langle x \times y, z \rangle$$
A 3-dimensional subspace $A \subset \text{Im}(\Theta)$ is *associative* if one of the (equivalent) holds:
A 3-dimensional subspace $A \subset \text{Im}(\mathcal{O})$ is *associative* if one of the (equivalent) holds:

- $A$ is closed under the cross product
A 3-dimensional subspace $\mathcal{A} \subset \text{Im}(\mathcal{O})$ is *associative* if one of the (equivalent) holds:

- $\mathcal{A}$ is closed under the cross product
- $\text{Re}(\mathcal{O}) \oplus \mathcal{A} \subset \mathcal{O}$ is a subalgebra
  (an associative subalgebra $\cong \mathbb{H}$)
A 3-dimensional subspace $\mathcal{A} \subset \text{Im}(\Theta)$ is *associative* if one of the (equivalent) holds:

- $\mathcal{A}$ is closed under the cross product
- $\text{Re}(\Theta) \oplus \mathcal{A} \subset \Theta$ is a subalgebra
  
  (an associative subalgebra $\simeq \mathbb{H}$)
- $\mathcal{A}$ can be oriented such that $\phi|_\mathcal{A} = d\text{vol}_\mathcal{A}$
  
  i.e. $\mathcal{A}$ is a calibrated subspace with respect to the calibration $\phi$
A 4-dimensional subspace $C \subset \text{Im}(\mathcal{O})$ is coassociative if one of the (equivalent) holds:
A 4-dimensional subspace $\mathcal{C} \subset \text{Im}(\mathcal{O})$ is coassociative if one of the (equivalent) holds:

- $\mathcal{C}^\perp$ is associative
A 4-dimensional subspace $\mathcal{C} \subset \text{Im} (\mathcal{O})$ is coassociative if one of the (equivalent) holds:

- $\mathcal{C}^\perp$ is associative
- For $x, y \in \mathcal{C}$, $x \times y \in \mathcal{C}^\perp$
Coassociative subspaces

A 4-dimensional subspace $\mathcal{C} \subset \text{Im}(\mathbb{O})$ is coassociative if one of the (equivalent) holds:

- $\mathcal{C}^\perp$ is associative
- For $x, y \in \mathcal{C}$, $x \times y \in \mathcal{C}^\perp$
- $\mathcal{C}$ is a calibrated subspace with respect to $\psi$: $\psi|_{\mathcal{C}} = d\text{vol}_{\mathcal{C}}$
A 4-dimensional subspace $C \subset \text{Im}(\Omega)$ is coassociative if one of the (equivalent) holds:

- $C^\perp$ is associative
- For $x, y \in C$, $x \times y \in C^\perp$
- $C$ is a calibrated subspace with respect to $\psi$: $\psi|_C = dvol_C$
- $\phi|_C = 0$
A $G_2$-structure on a 7-manifold is a reduction of structure of $TX$ to $G_2$. 

\[ \star \phi \in \Omega^4(X) \]
A $G_2$-structure on a 7-manifold is a reduction of structure of $TX$ to $G_2$

- each tangent space of $X$ has the structure of $\text{Im}(\mathbb{O})$
A $G_2$-structure on a 7-manifold is a reduction of structure of $TX$ to $G_2$

- each tangent space of $X$ has the structure of $\text{Im}(\mathcal{O})$

$X$ has:
A $G_2$-structure on a 7-manifold is a reduction of structure of $TX$ to $G_2$

- each tangent space of $X$ has the structure of $\text{Im}(\mathcal{O})$

$X$ has:

- a cross product $\times : TX \otimes TX \to TX$
A $G_2$-structure on a 7-manifold is a reduction of structure of $TX$ to $G_2$

- each tangent space of $X$ has the structure of $\text{Im}(\mathcal{O})$

$X$ has:

- a cross product $\times : TX \otimes TX \to TX$
- a Riemannian metric $g$
A $G_2$-structure on a 7-manifold is a reduction of structure of $TX$ to $G_2$

- each tangent space of $X$ has the structure of $\text{Im}(O)$

$X$ has:

- a cross product $\times : TX \otimes TX \rightarrow TX$
- a Riemannian metric $g$
- a 3-form $\phi \in \Omega^3(X)$
A $G_2$-structure on a 7-manifold is a reduction of structure of $TX$ to $G_2$

- each tangent space of $X$ has the structure of $\text{Im}(\Theta)$

$X$ has:

- a cross product $\times : TX \otimes TX \to TX$
- a Riemannian metric $g$
- a 3-form $\phi \in \Omega^3(X)$
- a 4-form $\psi = *\phi \in \Omega^4(X)$
A $G_2$-manifold is a Riemannian 7-manifold with holonomy in $G_2$. 

$G_2$-manifolds
A \textit{G}_2\text{-manifold} is a Riemannian 7-manifold with holonomy in \textit{G}_2.

- \textit{G}_2 is one of the exceptional holonomy groups in Berger’s classification.
A \textit{G}_2\text{-manifold} is a Riemannian 7-manifold with holonomy in \textit{G}_2

- \textit{G}_2 is one of the exceptional holonomy groups in Berger’s classification

so a \textit{G}_2\text{-manifold} \textit{X} is a Riemannian 7-manifold with a \textit{G}_2\text{-structure} such that

\[ \nabla \phi = 0 \]
A \textit{G}_2\text{-manifold} is a Riemannian 7-manifold with holonomy in \textit{G}_2

- \textit{G}_2 is one of the exceptional holonomy groups in Berger’s classification

so a \textit{G}_2\text{-manifold} \textit{X} is a Riemannian 7-manifold with a \textit{G}_2\text{-structure} such that

\[ \nabla \phi = 0 \]

equivalently: \textit{X} is a 7-manifold with a \textit{G}_2\text{-structure} such that

\[ d\phi = 0 \]
\[ d\psi = 0 \]
A 3-submanifold $\mathcal{A} \to X$ is an associative submanifold if:

1. The tangent spaces of $\mathcal{A}$ are associative subspaces of $T_X$.
2. $\mathcal{A}$ is a calibrated submanifold with respect to $\phi$.

Similarly, define coassociative submanifolds $\mathcal{C} \to X$:

1. $\mathcal{C}$ is a calibrated submanifold with respect to $\psi$.
2. $\mathcal{C}$ is coassociative if $\phi|_\mathcal{C} = 0$. 
A 3-submanifold $\mathcal{A} \to X$ is an associative submanifold if:

- the tangent spaces of $\mathcal{A}$ are associative subspaces of $TX$
A 3-submanifold $\mathcal{A} \to X$ is an associative submanifold if:
- the tangent spaces of $\mathcal{A}$ are associative subspaces of $TX$
- equivalently $\mathcal{A}$ is a calibrated submanifold wrt $\phi$
A 3-submanifold $\mathcal{A} \to X$ is an associative submanifold if:

- the tangent spaces of $\mathcal{A}$ are associative subspaces of $TX$
- equivalently $\mathcal{A}$ is a calibrated submanifold wrt $\phi$

Similarly define coassociative submanifolds $\mathcal{C} \to X$
A 3-submanifold $\mathcal{A} \to X$ is an associative submanifold if:
- the tangent spaces of $\mathcal{A}$ are associative subspaces of $TX$
- equivalently $\mathcal{A}$ is a calibrated submanifold wrt $\phi$

similarly define coassociative submanifolds $\mathcal{C} \to X$
- $\mathcal{C}$ is a calibrated submanifold wrt $\psi$
A 3-submanifold $\mathcal{A} \to X$ is an associative submanifold if:
- the tangent spaces of $\mathcal{A}$ are associative subspaces of $TX$
- equivalently $\mathcal{A}$ is a calibrated submanifold wrt $\phi$

similarly define coassociative submanifolds $\mathcal{C} \to X$
- $\mathcal{C}$ is a calibrated submanifold wrt $\psi$
- $\mathcal{C}$ is coassociative iff $\phi|_{\mathcal{C}} = 0$
First order deformations

Let $\mathcal{C} \rightarrow X$ be a \textit{compact} coassociative submanifold
Let $\mathcal{C} \rightarrow X$ be a *compact* coassociative submanifold then $N\mathcal{C} \simeq \bigwedge^2 C \oplus T^*\mathcal{C}$.
First order deformations

Let \( C \to X \) be a compact coassociative submanifold then \( NC \simeq \bigwedge^2_+ T^*C \):

\[ \nu \mapsto \iota_\nu \phi|_C \]
First order deformations

Let $C \to X$ be a compact coassciative submanifold then $NC \simeq \wedge^2_{+} T^*C$:

$$\nu \mapsto \iota_{\nu} \phi|_C$$

given a normal vector field $\nu$, we can deform $C$ in the direction $\nu$. 

First order deformations

Let $\mathcal{C} \to X$ be a compact coassociative submanifold then $N\mathcal{C} \cong \wedge^2_+ T^*\mathcal{C}$:

$$\nu \mapsto \iota_\nu \phi|_\mathcal{C}$$

given a normal vector field $\nu$, we can deform $\mathcal{C}$ in the direction $\nu$

**Theorem (McLean)**

A normal vector field $\nu$ represents a first order deformation through coassociative submanifolds iff

$$\iota_\nu \phi|_\mathcal{C} \text{ is closed},$$

hence a harmonic self-dual form.
No obstructions to extending a first order deformation to an actual family
No obstructions to extending a first order deformation to an actual family

we then have a smooth moduli space \( \mathcal{M} \) of deformations of \( C \) through
coassociative submanifolds
No obstructions to extending a first order deformation to an actual family
we then have a smooth moduli space $\mathcal{M}$ of deformations of $C$ through
coassociative submanifolds
the tangent space $T_C \mathcal{M}$ of $\mathcal{M}$ at $C$ is naturally isomorphic to $\mathcal{H}^2_+(C, \mathbb{R})$
No obstructions to extending a first order deformation to an actual family
we then have a smooth moduli space $\mathcal{M}$ of deformations of $C$ through
coassociative submanifolds
the tangent space $T_CM$ of $\mathcal{M}$ at $C$ is naturally isomorphic to $\mathcal{H}^2_+(C, \mathbb{R})$
$$\dim(\mathcal{M}) = b^2_+(C)$$
No obstructions to extending a first order deformation to an actual family
we then have a smooth moduli space $\mathcal{M}$ of deformations of $\mathcal{C}$ through coassociative submanifolds
the tangent space $T_{\mathcal{C}}\mathcal{M}$ of $\mathcal{M}$ at $\mathcal{C}$ is naturally isomorphic to $\mathcal{H}^2_+(\mathcal{C}, \mathbb{R})$

$$\dim(\mathcal{M}) = b^2_+(\mathcal{C})$$

notation: for $X \in T_{\mathcal{C}}\mathcal{M}$ let $\omega_X$ be the corresponding harmonic form
Let $X, Y \in T_{C\mathcal{M}}$

define a metric $g_{\mathcal{M}}$ on $\mathcal{M}$:
Let $X, Y \in T_{CM}$

define a metric $g_M$ on $\mathcal{M}$:

$$g_M(X, Y) = \int_C \omega_X \wedge \omega_Y$$

$$= \langle [\omega_X] \sim [\omega_Y], [C] \rangle$$

called the $L^2$ moduli space metric
For small enough deformations we can canonically identify cohomology of each submanifold with a fixed $C \in \mathcal{M}$.

Theorem: $\alpha$ is closed, so (locally) we have a function $u: M \to H^2(C, \mathbb{R})$ such that $\alpha = du$.

$u^*(X) = \alpha(X) = [\omega_X]$.
For small enough deformations we can canonically identify cohomology of each submanifold with a fixed $C \in \mathcal{M}$

get a (locally defined) $H^2(C, \mathbb{R})$-valued 1-form $\alpha$:

$$T\mathcal{M} \ni X \mapsto \alpha(X) = [\omega_X] \in H^2(C, \mathbb{R})$$
Local moduli space structure

For small enough deformations we can canonically identify cohomology of each submanifold with a fixed \( C \in \mathcal{M} \)

get a (locally defined) \( H^2(C, \mathbb{R}) \)-valued 1-form \( \alpha \):

\[
T\mathcal{M} \ni X \mapsto \alpha(X) = [\omega_X] \in H^2(C, \mathbb{R})
\]

**Theorem**

\( \alpha \) is closed, so (locally) we have a function

\[
u : \mathcal{M} \to H^2(C, \mathbb{R})
\]

such that \( \alpha = du \):

\[
u_*(X) = \alpha(X) = [\omega_X]
\]
Then we have

\[ g_M(X, Y) = \langle [\omega_X] \sim [\omega_Y], [C] \rangle \]
\[ = \langle u_*(X) \sim u_*(Y), [C] \rangle \]
Then we have

\[ g_{\mathcal{M}}(X, Y) = \langle [\omega_X] \sim [\omega_Y], [C] \rangle \]

\[ = \langle u_*(X) \sim u_*(Y), [C] \rangle \]

so the moduli space metric is the pull-back under \( u \) of the intersection form on \( H^2(C, \mathbb{R}) \)
Then we have

\[ g_M(X, Y) = \langle [\omega_X] \sim [\omega_Y], [C] \rangle \]
\[ = \langle u_*(X) \sim u_*(Y), [C] \rangle \]

so the moduli space metric is the pull-back under \( u \) of the intersection form on \( H^2(C, \mathbb{R}) \)

\( M \) is immersed as a maximal positive definite submanifold of \( H^2(C, \mathbb{R}) \).
Theorem

Let $X$ be compact and have holonomy $= G_2$
Compact Fibrations

**Theorem**

Let $X$ be compact and have holonomy $= G_2$

then any coassociative fibration $\pi : X \rightarrow B$ must degenerate

(i.e. $\pi$ can’t be a submersion everywhere)
Proof (1)

Assume $F \to X \to B$ is a non-degenerate fibration
Assume \( F \to X \to B \) is a non-degenerate fibration

\[ \text{Hol}(X) = G_2 \Rightarrow \pi_1(X) \text{ is finite} \]

suffices to take \( X \) simply connected

\[ B \text{ is a homotopy 3-sphere and } F \text{ is simply connected} \]
Assume $F \to X \to B$ is a non-degenerate fibration

$\text{Hol}(X) = G_2 \Rightarrow \pi_1(X)$ is finite

suffices to take $X$ simply connected

also suffices to take $F$ connected and $B$ simply connected
Assume $F \to X \to B$ is a non-degenerate fibration

$\text{Hol}(X) = G_2 \Rightarrow \pi_1(X)$ is finite

suffices to take $X$ simply connected

also suffices to take $F$ connected and $B$ simply connected

Then $B$ is a homotopy 3-sphere and $F$ is simply connected
Leray-Serre spectral sequence $\Rightarrow$ isomorphisms

\[ i^* : H^2(X, \mathbb{R}) \to H^2(F, \mathbb{R}) \]
\[ \pi^* : H^3(B, \mathbb{R}) \to H^3(X, \mathbb{R}) \]
Proof (2)

Leray-Serre spectral sequence $\Rightarrow$ isomorphisms

\[ i^* : H^2(X, \mathbb{R}) \to H^2(F, \mathbb{R}) \]
\[ \pi^* : H^3(B, \mathbb{R}) \to H^3(X, \mathbb{R}) \]

\[ [\phi] = c\pi^*([dvol_B]), \quad c \neq 0, \quad \int_B dvol_B = 1 \]
Proof (2)

Leray-Serre spectral sequence $\Rightarrow$ isomorphisms

\[ i^* : H^2(X, \mathbb{R}) \to H^2(F, \mathbb{R}) \]
\[ \pi^* : H^3(B, \mathbb{R}) \to H^3(X, \mathbb{R}) \]

\[ [\phi] = c\pi^*([dvol_B]), \quad c \neq 0, \quad \int_B dvol_B = 1 \]

then for a closed 4-form $\mu$ on $X$

\[ \int_X \mu \wedge \phi = c \int_F i^* \mu \]
Recall: for a $G_2$-manifold (with $b^1(X) = 0$) the pairing $H^2(X, \mathbb{R}) \otimes H^2(X, \mathbb{R}) \to \mathbb{R}$ given by

$$\int_X \alpha \wedge \beta \wedge \phi$$

is negative definite
Recall: for a $G_2$-manifold (with $b^1(X) = 0$) the pairing $H^2(X, \mathbb{R}) \otimes H^2(X, \mathbb{R}) \to \mathbb{R}$ given by

$$\int_X \alpha \wedge \beta \wedge \phi$$

is negative definite

hence the intersection form on $F$ is negative definite
Recall: for a $G_2$-manifold (with $b^1(X) = 0$) the pairing $H^2(X, \mathbb{R}) \otimes H^2(X, \mathbb{R}) \to \mathbb{R}$ given by

$$\int_X \alpha \wedge \beta \wedge \phi$$

is negative definite

hence the intersection form on $F$ is negative definite

(Donaldson): intersection form on $F$ is diagonal
(i.e. of form $\text{diag}(-1, -1, \ldots, -1)$)
Now $X$ is spin and $NF$ is trivial
Now $X$ is spin and $NF$ is trivial $\Rightarrow F$ is spin
Proof (4)

Now $X$ is spin and $NF$ is trivial $\Rightarrow F$ is spin
$\Rightarrow$ intersection form is even
Now $X$ is spin and $NF$ is trivial $\Rightarrow F$ is spin
$\Rightarrow$ intersection form is even $\Rightarrow b^2(F) = 0$
Proof (4)

Now $X$ is spin and $NF$ is trivial $\Rightarrow F$ is spin
$\Rightarrow$ intersection form is even $\Rightarrow b^2(F) = 0$

also recall for a $G_2$-manifold (again with $b^1(X) = 0$)

$$\int_X p_1(X) \wedge \phi < 0$$
Now $X$ is spin and $NF$ is trivial $\Rightarrow F$ is spin
$\Rightarrow$ intersection form is even $\Rightarrow b^2(F) = 0$
also recall for a $G_2$-manifold (again with $b^1(X) = 0$)

$$\int_X p_1(X) \wedge \phi < 0$$

but this is

$$c \int_F i^* p_1(X) = c \int_F p_1(F) = 0$$

by Hirzebruch signature theorem $\Rightarrow$ contradiction
What do the singularities look like?
What do the singularities look like?

as an example of what to expect we construct compact coassociative fibrations of $G_2$-structures with torsion ($d\phi \neq 0$)
What do the singularities look like?

as an example of what to expect we construct compact coassociative fibrations of $G_2$-structures with torsion ($d\phi \neq 0$)

one would hope that metric could be adjusted to produce genuine $G_2$-manifolds
Take a holomorphic symplectic fibration $\pi : M \to \mathbb{CP}^2$ of a Hyperkähler 8-manifold (e.g. the Hilbert scheme of an elliptic $K3$-fibration)
Take a holomorphic symplectic fibration $\pi : M \to \mathbb{CP}^2$ of a Hyperkähler 8-manifold (e.g. the Hilbert scheme of an elliptic $K3$-fibration)

the fibration degenerates over a curve $\Delta \subset \mathbb{CP}^2$
Take a holomorphic symplectic fibration $\pi : M \to \mathbb{CP}^2$ of a Hyperkähler 8-manifold (e.g. the Hilbert scheme of an elliptic $K3$-fibration)

the fibration degenerates over a curve $\Delta \subset \mathbb{CP}^2$

then $M$ is also a $\text{Spin}(7)$-manifold with 4-form

$$\Phi = \frac{1}{2} \omega^2_i + \frac{1}{2} \omega^2_j - \frac{1}{2} \omega^2_K$$
Take a holomorphic symplectic fibration $\pi : M \to \mathbb{CP}^2$ of a Hyperkähler 8-manifold (e.g. the Hilbert scheme of an elliptic $K3$-fibration) the fibration degenerates over a curve $\Delta \subset \mathbb{CP}^2$ then $M$ is also a $\text{Spin}(7)$-manifold with 4-form

$$\Phi = \frac{1}{2} \omega^2_i + \frac{1}{2} \omega^2_j - \frac{1}{2} \omega^2_K$$

and $\pi : M \to \mathbb{CP}^2$ is a fibration by Cayley 4-folds
Now take $S^3 \subset \mathbb{CP}^2$ that encloses a singularity of the curve.
Now take $S^3 \subset \mathbb{CP}^2$ that encloses a singularity of the curve
then $S^3 \cap \Delta$ is a smooth link
Now take $S^3 \subset \mathbb{CP}^2$ that encloses a singularity of the curve

then $S^3 \cap \Delta$ is a smooth link

e.g. around a singularity $x^3 + y^2 = 0$ get a trefoil knot
Now take $S^3 \subset \mathbb{CP}^2$ that encloses a singularity of the curve

then $S^3 \cap \Delta$ is a smooth link

e.g. around a singularity $x^3 + y^2 = 0$ get a trefoil knot

$\pi^{-1}(S^3)$ is an almost $G_2$-manifold ($d\phi \neq 0$, $d\psi = 0$) with coassociative fibration degenerating over $S^3 \cap \Delta$
Now take $S^3 \subset \mathbb{CP}^2$ that encloses a singularity of the curve

then $S^3 \cap \Delta$ is a smooth link

e.g. around a singularity $x^3 + y^2 = 0$ get a trefoil knot

$\pi^{-1}(S^3)$ is an almost $G_2$-manifold ($d\phi \neq 0$, $d\psi = 0$) with coassociative fibration degenerating over $S^3 \cap \Delta$

$\psi$ closed $\implies$ the fibres are minimal submanifolds, could we flow to a fibration of a $G_2$-manifold?
Let $\pi : X \to B$ be a coassociative fibration with compact fibres.
Theorem

Let $\pi : X \to B$ be a coassociative fibration with compact fibres. $B$ can be given a metric $g_B$ such that $\pi : X \to B$ is a Riemannian submersion iff the fibres are Hyperkähler (so either $T^4$ or $K3$).
Riemannian submersion case

Theorem

Let $\pi : X \to B$ be a coassociative fibration with compact fibres

$B$ can be given a metric $g_B$ such that $\pi : X \to B$ is a Riemannian submersion iff the fibres are Hyperkähler (so either $T^4$ or $K3$)

in this case $B$ is the moduli space of deformations and

$$g_B = \frac{1}{2\text{vol}(F)}g_{L^2}$$
Proof

Let $F = \pi^{-1}(b)$ be the fibre over $b \in B$
Proof

Let $F = \pi^{-1}(b)$ be the fibre over $b \in B$

pick a basis $e_1, e_2, e_3$ for $T_b B$, let $\tilde{e}_i$ be the horizontal lifts
Proof

Let $F = \pi^{-1}(b)$ be the fibre over $b \in B$

pick a basis $e_1, e_2, e_3$ for $T_bB$, let $\tilde{e}_i$ be the horizontal lifts

let $\omega_i = \iota_{\tilde{e}_i} \phi|_F$ be the corresponding harmonic forms, then

$$\omega_i \wedge \omega_j = 2g(\tilde{e}_i, \tilde{e}_j) d\text{vol}_F$$
Proof

Let $F = \pi^{-1}(b)$ be the fibre over $b \in B$

pick a basis $e_1, e_2, e_3$ for $T_b B$, let $\tilde{e}_i$ be the horizontal lifts

let $\omega_i = \iota_{\tilde{e}_i} \phi|_F$ be the corresponding harmonic forms, then

$$\omega_i \wedge \omega_j = 2g(\tilde{e}_i, \tilde{e}_j) d\text{vol}_F$$

follows that we can find Hyperkähler forms iff $\pi$ is a Riemannian submersion
1 Some dualities

2 $G_2$ geometry

3 Deformations of coassociative submanifolds

4 Coassociative fibrations

5 Semi-flat coassociative fibrations
A coassociative fibration $\pi : X \to B$ is \textit{semi-flat} if there is a $T^4$-action of isomorphisms such that the orbits are the fibres of $\pi$. $\pi$ is a Riemannian submersion and $B$ is the moduli space of deformations. (locally) we have the moduli space map $u : B \to H_2(T^4, \mathbb{R}) \cong \mathbb{R}^3$. 

David Baraglia (ANU)  Moduli of special Lagrangian and coassociative  July 19, 2010  33 / 39
Definition

A coassociative fibration $\pi : X \to B$ is semi-flat if

- there is a $T^4$-action of isomorphisms such that the orbits are the fibres of $\pi$.
Definition

A coassociative fibration $\pi : X \to B$ is *semi-flat* if

- there is a $T^4$-action of isomorphisms such that the orbits are the fibres of $\pi$

$\pi$ is a Riemannian submersion and $B$ is the moduli space of deformations
Definition

A coassociative fibration \( \pi : X \to B \) is semi-flat if

- there is a \( T^4 \)-action of isomorphisms such that the orbits are the fibres of \( \pi \)

\( \pi \) is a Riemannian submersion and \( B \) is the moduli space of deformations

(locally) we have the moduli space map

\[
u : B \to H^2(T^4, \mathbb{R}) \cong \mathbb{R}^{3,3}
\]
Semi-flat fibrations are locally constructed from the following data:
Semi-flat fibrations are locally constructed from the following data:
- an oriented 3-manifold $B$
Semi-flat fibrations are locally constructed from the following data:

- an oriented 3-manifold $B$
- $u : B \to H^2(T^4, \mathbb{R}) \cong \mathbb{R}^{3,3}$ that maps $TB$ to positive definite subspaces
Semi-flat fibrations are locally constructed from the following data:

- an oriented 3-manifold $B$
- $u : B \to H^2(T^4, \mathbb{R}) \cong \mathbb{R}^{3,3}$ that maps $TB$ to positive definite subspaces
- $\tau$ a positive constant (representing the fibre volume)
Give $B$ the metric

$$g_B = \frac{1}{2\tau} u^*(g_{3,3})$$
Give $B$ the metric

$$g_B = \frac{1}{2\tau} u^*(g_{3,3})$$

define a 3-form $\phi$ on $B \times \mathbb{R}^4/Z^4$
Give $B$ the metric

$$g_B = \frac{1}{2\tau} u^*(g_{3,3})$$

define a 3-form $\phi$ on $B \times \mathbb{R}^4/\mathbb{Z}^4$

$$\phi = d\text{vol}_B + du$$

where $u : B \to H^2(T^4, \mathbb{R})$ is thought of as a 2-form

$$u(b) = u_{ij}(b) dx^{ij}$$
Local form (3)

- $\phi$ is closed
Local form (3)

- $\phi$ is closed
- $\phi$ has the correct algebraic form
Local form (3)

- $\phi$ is closed
- $\phi$ has the correct algebraic form

**Theorem**

$$d\psi = 0 \iff u \text{ is a harmonic map}$$

i.e. $u : B \to H^2(T^4, \mathbb{R})$ is a minimal immersion
Local form (3)

- $\phi$ is closed
- $\phi$ has the correct algebraic form

**Theorem**

\[ d\psi = 0 \text{ iff } u \text{ is a harmonic map} \]

i.e. \( u : B \to H^2(T^4, \mathbb{R}) \) is a minimal immersion

*all semi-flat fibrations have this form locally*
Suppose that $u : B \to \mathbb{R}^{3,3}$ is conical:

$$B = (0, \infty) \times \Sigma$$

$$g_B = dr^2 + r^2 g_\Sigma$$
Suppose that $u : B \to \mathbb{R}^{3,3}$ is conical:

$$B = (0, \infty) \times \Sigma$$

$$g_B = dr^2 + r^2 g_{\Sigma}$$

this corresponds to a (positive definite) minimal surface in the quadric

$$Q = \{ x \in \mathbb{R}^{3,3} | \langle x, x \rangle = 1 \}$$
A particular class of minimal surfaces into $Q$ correspond to the equations:

\[
\begin{align*}
    u_{zz} &= -e^v - u - e^u, \\
    v_{zz} &= q e^v - v + e^v - u.
\end{align*}
\]
Affine Toda equations

A particular class of minimal surfaces into $Q$ correspond to the equations:

\[ u_{zz} = -e^{v-u} - e^u, \]
\[ v_{zz} = q\bar{q}e^{-v} + e^{v-u}. \]

where $q$ is a holomorphic cubic differential

- this is a set of affine Toda equations
THANK YOU