Moduli of special Lagrangian and coassociative submanifolds

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Why look at special Lagrangian / coassociative submanifolds?
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Introduction

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- Dualities in M-theory: SYZ-like conjectures involving coassociative fibrations of $G_2$-manifolds.
Let $(X, g)$ be a Riemannian manifold
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**Definition**

A compact oriented submanifold \(S \to X\) is called a *minimal submanifold* if it is a stationary point for the volume functional

\[
\text{vol}(S) = \int_S d\text{vol}_S
\]
A classical result

**Theorem**

Let $X$ be Kähler. Any compact complex submanifold of $X$ is minimal.
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Ex: A non-singular algebraic variety in $\mathbb{CP}^n$ is a minimal submanifold.

Harvey and Lawson found a generalization of this result using calibrations.
Calibrations

Let \((X, g)\) be a Riemannian manifold.
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A *calibration* \(\phi\) on \(X\) is a \(p\)-form such that

- \(\phi\) is closed: \(d\phi = 0\),
- for any \(x \in X\) and oriented \(p\)-dimensional subspace \(V \subseteq T_x X\), we have \(\phi|_V = \lambda d\text{vol}\), where \(\lambda \leq 1\) and \(d\text{vol}\) is the volume form on \(V\) with respect to \(g\).
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Subspaces such that \(\lambda = 1\) are called *calibrated subspaces*. 
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Subspaces such that \(\lambda = 1\) are called *calibrated subspaces.*

An oriented \(p\)-submanifold \(S \subseteq X\) is a *calibrated submanifold* of \(X\) with respect to \(\phi\) if the tangent spaces of \(S\) are calibrated subspaces: \(\phi|_S = dvol_S\).
Calibrations

**Theorem**

Let $S$ be a compact calibrated submanifold. $S$ has minimal volume amongst all submanifolds representing the same homology class.
Calibrations

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Proof.

Let $S'$ be a compact submanifold with $[S] = [S']$. Then since $\phi$ is closed

$$\text{vol}(S) = \int_S \phi = \int_{S'} \phi \leq \int_{S'} d\text{vol}_{S'} = \text{vol}(S').$$
Why is this useful?
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The minimal submanifold equation $f : S \to X$ is second order in $f$.

The calibrated submanifold condition $f^* \phi = dvol_S$ is first order in $f$. 
Some examples

\((X, \omega)\) Kähler, then \(\omega^k\) is a calibration \(\implies\) complex submanifolds
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Calabi-Yau: special Lagrangians (soon)
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Special Lagrangian and coassociative have a nice deformation theory (unobstructed). Associative and Cayley do not (obstructed). (See: McLean)
A Calabi-Yau manifold is a Riemannian manifold with holonomy in $SU(n)$. On a Kähler manifold $X$ with trivial canonical bundle ($K = \mathbb{C}^n, 0$), every Kähler class admits a unique Calabi-Yau metric.
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On a Kähler manifold $X$ with trivial canonical bundle ($K = \bigwedge^{n,0} T^*X$), every Kähler class admits a unique Calabi-Yau metric.
SU(n)-structures

Reduction of structure to SU(n) can be defined using only

- A non-degenerate 2-form $\omega$
- A complex $n$-form $\Omega = \Omega_1 + i\Omega_2$ which is locally decomposable: $\Omega = \theta_1 \wedge \cdots \wedge \theta_n$

such that:

$$\omega \wedge \Omega = 0, \quad \Omega \wedge \Omega = \omega_n \text{ or } i\omega_n,$$

$\omega(I, \cdot)$ is positive with respect to the almost complex structure determined by $\Omega$. 

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$SU(n)$-structures
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In this case there is a torsion free $SU(n)$-connection $\nabla$ such that $\nabla \omega = 0$, $\nabla \Omega = 0$.

Precisely the requirement for a Calabi-Yau manifold.
The form $\Omega_1 = \text{Re}(\Omega)$ is a calibration.

**Definition**

A submanifold $L \rightarrow X$ of a Calabi-Yau manifold is *special Lagrangian* if it is a calibrated submanifold with respect to $\Omega_1$. 

Note: can replace $\Omega$ by $e^{-i\theta}\Omega$ and $\Omega_1$ by $\cos(\theta)\Omega_1 + \sin(\theta)\Omega_2$.

Equivalent condition: $\omega|_L = 0$, $\Omega_2|_L = 0$ (hence the "Lagrangian" part of the name).
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special Lagrangians are much more rigid - we will see that compact ones have a finite dimensional moduli space of deformations of dimension \( b^1 \).

special Lagrangian tori exist in \( b^1(T^n) = n \) dimensional families - just the right number for a torus fibration
Let $L \rightarrow X$ be a *compact* special Lagrangian.
First order deformations

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given a normal vector field $X$, we can deform $L$ in the direction $X$

**Theorem (McLean)**

*A normal vector field $X$ represents a first order deformation through special Lagrangian submanifolds iff*

$$i_X \omega |_L$$ *is a harmonic 1-form on $L$.***
Moduli space of deformations

No obstructions to extending a first order deformation to an actual family
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notation: for $X \in T_L\mathcal{M}$ let $\theta_X = i_X \omega|_L$ be the corresponding harmonic form
$\mathcal{M}$ has a natural metric:
\( \mathcal{M} \) has a natural metric:

Let \( X, Y \in T_L \mathcal{M} \)

define \( g_\mathcal{M} \) on \( \mathcal{M} \):

\[
g_\mathcal{M}(X, Y) = \int_L \theta_X \wedge \star \theta_Y
\]

called the \( L^2 \) moduli space metric
Local moduli space structure

For small enough deformations we can canonically identify cohomology of each submanifold with a fixed $L \in \mathcal{M}$. 

Theorem

$\alpha$ is closed, so (locally) we have a function $u: \mathcal{M} \to H_1(L, \mathbb{R})$ such that $\alpha = du: u^*(x) = \alpha(x) = [\theta x]$. 

These are natural local affine coordinates $u_1, \ldots, u_b(L)$ on $\mathcal{M}$. 

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$T \mathcal{M} \ni X \mapsto \alpha(X) = [\theta_X] \in H^1(L, \mathbb{R})$
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$X$ is a first order deformation if and only if $i_X \Omega_2$ is a harmonic $(n-1)$-form. In fact $i_X \Omega = \star \theta_X$. 
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By same reasoning we get local affine coordinates $\nu: \mathcal{M} \rightarrow H^{n-1}(L, \mathbb{R})$. 
We have two sets of affine coordinates:

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Combine them: \( F = (u, v) : \mathcal{M} \to H^1(L, \mathbb{R}) \oplus H^{n-1}(L, \mathbb{R}) \).
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Why do this?

The space \( V = H^1(L, \mathbb{R}) \oplus H^{n-1}(L, \mathbb{R}) \) has an obvious \((n, n)\)-metric \( \langle , \rangle \) and symplectic structure \( \omega \):

\[
\langle (a, b), (c, d) \rangle = \frac{1}{2} \int_L a \wedge d + b \wedge c
\]

\[
\omega((a, b), (c, d)) = \int_L a \wedge d - b \wedge c.
\]
Local moduli space structure

**Theorem (Hitchin)**

\[ F : \mathcal{M} \to H^1(L, \mathbb{R}) \oplus H^{n-1}(L, \mathbb{R}) \] sends \( \mathcal{M} \) to a Lagrangian submanifold. Moreover the natural \( L^2 \) metric on \( \mathcal{M} \) is the induced metric.
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Theorem (Hitchin)

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Let \( u_1, \ldots, u_m \) be coords for \( H^1(L, \mathbb{R}) \), \( (m = b^1(L)) \).
Let \( v_i, \ldots, v_m \) be dual coordinates for \( H^{n-1}(L, \mathbb{R}) \).
The \((n, n)\) metric has form \( \sum_i du_i dv_i \).
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\( \mathcal{M} \) is a Lagrangian submanifold, so locally there is a function \( \phi \) such that \( v_i = \frac{\partial \phi}{\partial u_i} \).
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\( M \) is a Lagrangian submanifold, so locally there is a function \( \phi \) such that \( v_i = \frac{\partial \phi}{\partial u_i} \).

Therefore the \( L^2 \)-metric looks like

\[ g_M = \sum_{i,j} \frac{\partial^2 \phi}{\partial u_i \partial u_j} du_i du_j \]
\( \mathcal{M} \) is a Lagrangian, but is it in some sense “special”? 

Let

\[
W_1 = du_1 \wedge du_2 \wedge \cdots \wedge du_m
\]

\[
W_2 = dv_1 \wedge \cdots \wedge dv_m.
\]

Then some linear combination

\[
c_1 W_1 + c_2 W_2
\]

vanishes on \( \mathcal{M} \) if and only if \( \phi \) obeys the Monge-Ampère equation:

\[
\det(\text{Hess}(\phi)) = \text{const}
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However this does not hold for all moduli spaces. More on this soon.
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Conversely:

**Theorem (Bryant)**

Let $g$ be a metric on $T^n$ such that every non-zero harmonic 1-form is non-vanishing. Then $(T^n, g)$ appears as a fibre in a special Lagrangian fibration. This is a local result: the total space need not be compact or complete.
Let $\mathcal{M}$ be the moduli spaces of deformations of $L$. Consider the enlarged moduli space

$$\mathcal{M}^c = \mathcal{M} \times H^1(L, \mathbb{R}/\mathbb{Z})$$

special Lagrangians with flat $U(1)$-connections on them.
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special Lagrangians with flat $U(1)$-connections on them.

Then $T_{\mathcal{L}, \nabla} \mathcal{M}^c \simeq H^1(L, \mathbb{R}) \oplus H^1(L, \mathbb{R})$. Put the obvious almost complex structure and metric.
Monge-Ampère revisited

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Then $T_{L, \nabla} \mathcal{M}^c \simeq H^1(L, \mathbb{R}) \oplus H^1(L, \mathbb{R})$. Put the obvious almost complex structure and metric.

**Theorem**

$\mathcal{M}^c$ is Kähler. The fibres of $\mathcal{M}^c \to \mathcal{M}$ are Lagrangian.
If $u_1, \ldots, u_m$ are the local affine coords on $\mathcal{M}$ and $x_1, \ldots, x_m$ corresponding coords on the fibres, let

$$\tilde{\Omega} = d(u_1 + ix_1) \wedge \cdots \wedge d(u_n + ix_n)$$
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Theorem

\[\tilde{\Omega}\] together with the Kähler structure on $\mathcal{M}^c$ defines a Calabi-Yau structure if and only if $\phi$ obeys the Monge-Ampère equation.

Solutions of the Monge-Ampère equation define a special Lagrangian fibration with flat fibres (semi-flat). Converse also true (up to monodromy).
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If we start with $X$ a semi-flat fibration then $\mathcal{M}^c$ deserves to be called the mirror of $X$. 

THANK YOU