(II) The heat kernel of the Dirac operator

For a Dirac operator D on a Clifford module E, its square is a Laplace type operator:

$$D^2\psi = \sum_{ij} c(e_i)\nabla_i c(e_j)\nabla_j \psi$$

$$= \sum_{ij}\left(c(e_i)c(e_j)\nabla_i\nabla_j\psi + c(e_i)c(\nabla_i e_j)\nabla_j\psi\right)$$

$$= -\sum_i \nabla_i^2\psi + \sum_{ij} c(e_i)c(\nabla_i e_j)\nabla_j\psi$$

In fact, this is commonly used as a definition for general Dirac operators.

Because of this, the <u>heat equation</u>

$$(*)\quad \begin{cases} \frac{\partial}{\partial t}\psi(t) + D^2\psi(t) = 0 \\ \psi(0) = \psi_0 \end{cases}$$

is well-posed on closed manifolds X.

One has a solution operator, suggestively written

$$e^{-tD^2} : L^2(X,E) \longrightarrow L^2(X,E)$$

such that $\psi(t) := e^{-tD^2}\psi_0$ solves $(*)$.

<u>Thm.</u> There is a smooth section p of the vector bundle $E^* \boxtimes E$ with fibers

$$(E^* \boxtimes E)_{(t,x,y)} = \mathrm{Hom}(E_x, E_y)$$

over $(0,\infty) \times X \times X$ such that

$$(e^{-tD^2}\psi_0)(y) = \int_X p_t(x,y)\,\psi_0(x)\,dx$$

for all $\psi_0 \in L^2(X,E)$.

To roughly see how to construct this <u>heat kernel</u> $p_t(x,y)$, observe that by our assumptions, D^2 is a self-adjoint Laplace type operator. Therefore, elliptic theory implies that it has an L^2-orthonormal basis of eigenvectors

$$\varphi_1, \varphi_2, \ldots \in C^\infty(X, E)$$

for eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \nearrow \infty .$$

Then it is clear that <u>formally</u>, one has

$$p_t(x,y) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \varphi_k^*(x) \otimes \varphi_k(y).$$

Namely, if $\psi = \sum_{k=1}^{\infty} a_k \varphi_k$, then

$$\int_X p_t(x,y)\psi(x)\,dx = \sum_{k=1}^{\infty} a_k e^{-t\lambda_k} \varphi_k .$$

This equals ψ for $t=0$, and

$$\left(\frac{\partial}{\partial t} + D^2\right)\psi = \sum_{k=1}^{\infty} a_k \left(\frac{d}{dt} e^{-t\lambda_k} \varphi_k + \lambda_k e^{-t\lambda_k} \varphi_k\right)$$

$$= 0 .$$

To see that $p_t(x,y)$ is in fact a smooth function, one can use the following black box.

Thm. (i) There exist $c, C > 0$ such that
$$c k^{2/n} \leq \lambda_k \leq C k^{2/n}.$$
(ii) There exist $C, m > 0$ such that
$$\|\varphi_k\| \leq C k^m.$$

To apply this, use some Sobolev theory. The spaces $H^k(X,E)$ are defined by
$$\|\psi\|_{H^k}^2 = \sum_{e=0}^{k} \|\nabla^e \psi\|_{L^2}^2.$$

One needs two facts:

(1) $\|\psi\|_{D,k}^2 := \|\psi\|_{L^2}^2 + \|D^k\psi\|_{L^2}^2$ (elliptic estimates) gives an equivalent norm.

(2) $H^{k+[\frac{n}{2}]+1}(X,E) \subseteq C^k(X,E)$. (Sobolev embedding).

The smoothness of $p_t(x,y)$ means that e^{-tD^2} is trace-class for $t>0$, with trace

$$\mathrm{Tr}(e^{-tD^2}) = \int_X \mathrm{tr}\, p_t(x,x)\, dx$$

How does this relate to index theory?

Remember that $E = E^+ \oplus E^-$ and that D is odd, i.e. it maps $E^\pm$ to $E^\mp$.

<u>Thm</u> (McKean-Singer). For any $t>0$,

$$\begin{aligned} \mathrm{ind}(D) &= \mathrm{Str}(e^{-tD^2}) \\ &= \mathrm{Tr}(e^{-tD^-D^+} - e^{-tD^+D^-}). \end{aligned}$$

<u>Proof 1</u>: We have

$$\mathrm{Tr}(e^{-tD^-D^+} - e^{-tD^+D^-}) = \sum_{k=1}^{\infty} (e^{-t\lambda_k^+} - e^{-t\lambda_k^-}).$$

Suppose that

$$D^-D^+\varphi = \lambda\varphi \Rightarrow D^+D^-D^+\varphi = \lambda D^+\varphi$$

Hence $D^+\varphi$ is an eigenvector of D^+D^- with eigenvalue λ. We obtain a map

$$D^+ : \mathrm{Eig}(\lambda, D^-D^+) \longrightarrow \mathrm{Eig}(\lambda, D^+D^-)$$

Similarly, if

$$D^+D^-\psi = \lambda\psi \implies D^-D^+D^-\psi = \lambda D^-\psi,$$

hence

$$D^- : \mathrm{Eig}(\lambda, D^+D^-) \longrightarrow \mathrm{Eig}(\lambda, D^-D^+).$$

We obtain that

(1) D^+D^- and D^-D^+ have the same eigenvalues.

(2) For $\lambda \neq 0$, D^+ provides an isomorphism between the eigenspaces with inverse $\frac{1}{\lambda}D^-$.

Hence all the contributions in the supertrace corresponding to non-zero modes cancel, and

$$\sum_{k=1}^{\infty}\left(e^{-t\lambda_k^+} - e^{-t\lambda_k^-}\right) = \sum_{\lambda_k^+=0} \overbrace{e^{-t\lambda_k^+}}^{=1} - \sum_{\lambda_k^-=0} \overbrace{e^{-t\lambda_k^-}}^{=1}$$

$$= \dim\ker(D^-D^+) - \dim\ker(D^+D^-).$$

Finally, notice that

$$\ker D^- D^+ = \ker D^+, \quad \ker D^+ D^- = \ker D^-$$

The inclusion „$\supseteq$" is clear. For the other, notice that if $D^- D^+ \varphi = 0$, then

$$0 = \langle \varphi, D^- D^+ \varphi \rangle_{L^2} = \langle \underbrace{(D^-)^*}_{= D^+} \varphi, D^+ \varphi \rangle_{L^2}$$
$$= \| D^+ \varphi \|^2_{L^2},$$

hence $D^+ \varphi = 0$. $\square$

<u>Proof 2.</u> We use that the supertrace satisfies $\mathrm{Str}\,[A, B] = 0$ for the <u>super-commutator</u>

$$[A, B] = AB - (-1)^{|A||B|} BA.$$

Now

$$\frac{d}{dt} \mathrm{Str}\left(e^{-tD^2}\right) = \mathrm{Str}\left(D^2 e^{-tD^2}\right)$$
$$= \mathrm{Str}\left([D, De^{-tD^2}]\right) = 0,$$

as the Dirac operator is odd.

The proof now follows from the fact that

$$\lim_{t\to\infty} e^{-tD^2} = \Pi,$$

the projection onto the kernel of D^2, which is the same as the kernel of D, and

$$\begin{aligned} \operatorname{Str} \Pi &= \operatorname{Tr} \Pi^+ - \operatorname{Tr} \Pi^- \\ &= \dim \ker D^+ - \dim \ker D^- \\ &= \operatorname{ind}(D). \end{aligned} \qquad \square$$

Having established that

$$\begin{aligned} \operatorname{ind}(D) &= \operatorname{Str}(e^{-tD^2}) \\ &= \int_X p_t(x,x)\,dx \end{aligned}$$

for any $t>0$, we have in particular

$$\operatorname{ind}(D) = \lim_{t\to 0} \int_X p_t(x,x)\,dx.$$

It turns out that it is worthwhile to study the short-time asymptotics of $p_t(x,x)$. Namely, we have the following.

<u>Thm.</u> The heat kernel has the asymptotic expansion as $t \to 0$

$$p_t(x,x) \sim (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j a_j(x)$$

uniformly for $x \in M$, meaning that for each $T>0$ and each N, there exists $C>0$ such that

$$\left| p_t(x,x) - (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{N} t^j a_j(x) \right| \leq C t^{N-\frac{n}{2}+1}$$

for each $t \in (0,T]$ and each $x \in X$. The $a_j(x)$ are uniquely determined and depend on the curvature and at most j derivatives of it at x.

Hence we have the asymptotic expansion as $t \to 0$

$$\int_X \operatorname{str} p_t(x,x)\,dx \sim (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} \int_X \operatorname{str} a_j(x)\,dx.$$

On the other hand, the left hand side equals $\operatorname{ind}(D)$, hence is constant. We obtain that $\operatorname{ind}(D)$ equals the constant term in the asymptotic expansion.

<u>Corollary</u>. We have

$$\operatorname{ind}(D) = (4\pi)^{-\frac{n}{2}} \int_X \operatorname{str} a_{n/2}(x)\,dx.$$

The goal will then be to identify the term $\operatorname{str} a_{n/2}(x)$.

We now discuss how to obtain the asymptotic expansion.

The heat kernel can be exactly computed in $\mathbb{R}^n$; it is

$$\gamma_t(x,y) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d(x,y)^2}{4t}\right).$$

This function makes sense on any Riem. manifold with $d(x,y)$ being the Riemannian distance function; it is smooth on the set

$$X \bowtie X := \{(x,y) \in X \times X \mid \text{there ex. a unique minimizing geodesic between } x \text{ and } y\}.$$

Now we make the ansatz

$$p_t(x,y) = \gamma_t(x,y) \sum_{j=0}^{\infty} t^j a_j(x,y)$$

with $a_j(x,y) \in C^\infty(X \bowtie X, E^* \boxtimes E)$. We now derive equations on the a_j that needed to be fulfilled in order to satisfy the heat equation formally.

First one calculates

$$\frac{\partial}{\partial t}\gamma_t(x,y) = \gamma_t(x,y)\left(\frac{d(x,y)^2}{4t^2} - \frac{n}{2t}\right)$$

$$\mathrm{grad}_y\,\gamma_t(x,y) = \gamma_t(x,y)\left(-\frac{\mathrm{grad}_y\, d(x,y)^2}{4t}\right)$$

$$\Delta_y\,\gamma_t(x,y) = \gamma_t(x,y)\left(-\frac{d(x,y)^2}{4t^2} + \frac{\Delta_y d(x,y)^2}{4t}\right).$$

Therefore, setting $\mathcal{R} = \frac{1}{2}\mathrm{grad}_x\, d(x,y)^2$

$$\left(\frac{\partial}{\partial t} + D_y^2\right)\gamma_t(x,y)\sum_{j=0}^{\infty} t^j a_j(x,y)$$

$$= \gamma_t(x,y)\sum_{j=0}^{\infty}\Big[-\frac{n}{2}t^{j-1}a_j(x,y) + j\,t^{j-1}a_j(x,y)$$

$$-\frac{1}{2}t^{j-1}\nabla_{\mathcal{R}}\, a_j(x,y) + t^{j-1}\frac{\Delta_y d(x,y)^2}{4}a_j(x,y)$$

$$+\,t^j D^2 a_j(x,y)\Big]$$

Shifting an index, this equals the following.

$$\gamma_t(x,y)\sum_{j=0}^{\infty} t^j\Big[\frac{1}{2}\nabla_{\mathcal{R}} a_j(x,y) + \Big(\frac{\Delta_y d(x,y)^2}{4} + j - \frac{n}{2}\Big) a_j(x,y) + D_y^2 a_{j-1}(x,y)\Big].$$

Requiring this to vanish, we obtain the recursive transport equations

$$\nabla_{\mathcal{R}} a_0(x,y) + \Big(\frac{\Delta_y d(x,y)^2}{2} - n\Big) a_0(x,y) = 0.$$

$$\nabla_{\mathcal{R}} a_{j+1}(x,y) + \Big(\frac{\Delta_y d(x,y)^2}{2} + 2j - n\Big) a_{j+1}(x,y) = -2 D_y^2 a_j(x,y).$$

Fixing x, these equations reduce to ODE's along the geodesics emanating from x. As the radial vector field $\mathcal{R}$ in normal coordinates around x has the form

$$\mathcal{R} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i},$$

these ODE's are singular at the starting point. It turns out that in order to obtain smooth solutions for the $a_j(x,y)$, one needs to prescribe a single initial condition, namely the value for $a_0(x,x)$.

When taking $a_0(x,x) = \mathrm{id}_{E_x}$, one obtains unique solutions given recursively by

$$a_0(x,y) = \mathcal{J}(x,y)^{-1/2} P(x,y)$$

$$a_{j+1}(x,y) = -\int_0^1 s^{j+1} P(x,\gamma(s))\, a_j(\gamma(s),y)\, ds.$$

Here $P(x,y)$ is the parallel transport along the shortest geodesic from x to y (denoted by $\gamma(s)$ in the second equation), and

$$\mathcal{J}(x,y) = \left|\det\left(d\exp_x\big|_{\exp_x^{-1}(y)} : T_x X \to T_y X\right)\right|$$

is the Jacobian determinant of the Riem. exponential map.

This determines the coefficients $a_j(x,y)$. There are now different ways to prove that this formal solution is indeed asymptotic to the real solution. This uses essentially uniqueness of solutions of the heat equation.

(III) Getzler rescaling

Remember that our goal is to find a more explicit formula for the term

$$\int_X \mathrm{str}\, a_{\frac{n}{2}}(x,x) = \mathrm{ind}\,(\mathcal{D}).$$

Theoretically, one could just recursively solve the transport equations. However, that gets more and more complicated with increasing n and is not feasible.
We therefore follow the footsteps of Getzler to find a more explicit solution.