(I) Clifford algebras and Dirac operators.

Let V be a Euclidean vector space.

Def. The <u>Clifford algebra</u> $Cl(V)$ is the algebra generated by the elements of V, subject to

$$v \cdot w + w \cdot v = -2\langle v, w\rangle \cdot 1.$$

If $e_1, \ldots, e_n$ is an ON-basis of V, we have

$$e_i \cdot e_j = -e_j \cdot e_i, \qquad e_i^2 = -1.$$

$\Rightarrow$ $e_{i_1} \cdots e_{i_k}$, $i_1 < \cdots < i_k$, $0 \leq k \leq \dim(V)$

is a vector space basis of $Cl(V)$

$\Rightarrow$ $\dim(Cl(V)) = 2^n$.

Have the <u>symbol map</u>

$$\sigma : Cl(V) \longrightarrow \Lambda V, \quad e_{i_1} \cdots e_{i_k} \mapsto e_{i_1} \wedge \ldots \wedge e_{i_k}$$

and its inverse, the <u>quantization map</u>

$$q : \Lambda V \longrightarrow Cl(V)$$

$$v_1 \wedge \dots \wedge v_k \longmapsto \frac{1}{k!} \sum_{\tau \in S_k} v_{\tau_1} \cdot \dots \cdot v_{\tau_k} .$$

If X is a Riemannian manifold, we obtain the algebra bundles

$$Cl(X)_x := Cl(T_x X)$$

$$\mathbb{C}l(X)_x := Cl(T_x X) \otimes \mathbb{C} .$$

<u>Def.</u> Let X be an even-dimensional, oriented, closed Riemannian manifold. A (self-adjoint) Clifford module over X is a $\mathbb{Z}_2$-graded Hermitean vector bundle (E, h) over X with compatible connection ∇^E together with a bundle map

$$c : TX \longrightarrow \operatorname{End}^-(E)$$

such that $\forall\ v, w \in TX,\ e, f \in E$

(1) $c(v)c(w) + c(w)c(v) = -2\langle v,w\rangle \mathrm{id}_E$

(2) $h(c(v)e, f) = -h(e, c(v)f)$

(3) $\nabla^E_v(c(w)e) = c(\nabla^{TX}_v w)e + c(w)\nabla^E_v e.$

(Here, v, w, e, f need to be sections.)

In particular, this gives each E_x the structure of a $\mathbb{C}l(T_xX)$-module.

<u>Examples.</u>

1) $E = \Lambda TX \otimes \mathbb{C}$,

$$c(v)\omega = v^\flat \wedge \omega - v \lrcorner \omega.$$

Two possible gradings:

(a) $\Lambda TX = \Lambda^{ev} TX \oplus \Lambda^{odd} TX$

(b) $\Lambda TX = \Lambda^{+} TX \oplus \Lambda^{-} TX$

(Eigenspaces of Hodge-$*$)

2) If E is a Clifford module, (W, h_W, ∇^W) a graded geometric vector bundle, the $E \otimes W$ is a Clifford module:

- $(E \otimes W)^{\pm} = E^{+} \otimes W^{\pm} \oplus E^{-} \otimes W^{\mp}$
- $c_{E \otimes W}(v) = c_E(v) \otimes \mathrm{id}_W$
- $\nabla^{E \otimes W}$ = tensor product conn.

<u>Thm.</u> Let E be a Clifford module, and let F^E be its curvature. Then there is a unique decomposition

$$F^E = R^{\Sigma} + F^{E/\Sigma},$$

where $F^{E/\Sigma}(v, w) \in \mathrm{End}_{\mathcal{Cl}(x)}(E)$. Explicitly,

$$R^{\Sigma}(v, w) = \frac{1}{4} \sum_{kl} \langle R(e_i, e_j) v, w \rangle c(e_i) c(e_j).$$

$F^{E/\Sigma}$ is called <u>twisting curvature</u> of E.

Notice that in general, we have

$$\mathrm{End}(E) = \mathbb{C}\mathrm{l}(X) \otimes \mathrm{End}_{\mathbb{C}\mathrm{l}(X)}(E).$$

<u>Proof.</u> Define $F^{E/\Sigma} := F^E - R^\Sigma$

$$\nabla_v^E \nabla_w^E (c(x)e)$$

$$= \nabla_v^E \left(c(\nabla_w x)e + c(x)\nabla_w^E e \right)$$

$$= c(\nabla_v \nabla_w x)e + c(\nabla_w x)\nabla_v^E e$$

$$+ c(\nabla_v x)\nabla_w^E e + c(x)\nabla_v^E \nabla_w^E e$$

$$F^E(v,w)c(x)e = c(R(v,w)x)e + c(x)F^E(v,w)e$$

$$\Rightarrow [F^E(v,w), c(x)] = c(R(v,w)x)$$

Now $c(R(v,w)x) = [R^\Sigma(v,w), c(x)]$ $\square$

Def. X is <u>spin^c</u> if there exists a Clifford module Σ^c such that $\mathbb{C}\ell(X) = \mathrm{End}(\Sigma^c)$, in other words, we have $\mathrm{End}_{\mathbb{C}\ell(X)}(\Sigma^c) = \mathbb{C}\cdot id$.

For n even, we have

$$\mathbb{C}\ell(\mathbb{R}^n) \cong \mathrm{Mat}_{2^{n/2}}(\mathbb{C})$$

hence locally, we can always set $\Sigma = \mathbb{C}^{2^{n/2}}$.

$\Rightarrow$ Being spin^c is a topological condition.

Spin manifolds (to be defined later) have the additional property that the twisting curvature $F^{\Sigma^c/\Sigma}$ vanishes (in particular, they are spin^c).

Examples.

(1) All 4-dim'l, oriented, closed manifolds are spin^c.

(2) Let (M,g) be an almost-complex manifold. Set

$$TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$$

with

$$T^{1,0}X = \{ v \otimes 1 - Jv \otimes i \mid v \in TX \}$$

$$T^{0,1}X = \{ v \otimes 1 + Jv \otimes i \mid v \in TX \}.$$

Set

$$\Sigma^c := \Lambda_{\mathbb{C}}(T^{1,0}X),$$

with

$$c(v)\,\omega = \sqrt{2}\, v \wedge \omega \quad \text{if } v \in T^{1,0}X$$

$$c(v)\,\omega = \sqrt{2}\, \bar{v} \lrcorner\, \omega \quad \text{if } v \in T^{0,1}X.$$

This gives a spin^c-structure.

Def. The <u>Dirac operator</u> of a Clifford module E is given by

$$D^E = \sum_{i=1}^n c(e_i)\nabla^E_{e_i}.$$

This is an elliptic, formally self-adjoint Dirac type operator of first order. It is odd,

$$D \mathrel{\hat{=}} \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

where

$$D^\pm : \Gamma(X, E^\pm) \longrightarrow \Gamma(X, E^\mp)$$

<u>Analytic blackbox</u>: $D^\pm$ have finite-dimensional kernel, hence we can consider

$$\operatorname{ind}(D^E) = \dim\ker D^+ - \dim\ker D^-.$$

Example.

(1) $E = \Lambda^{ev} T^*X \oplus \Lambda^{odd} T^*X$

$\Longrightarrow \operatorname{ind} D^E = \chi(X).$

(2) $E = \Lambda^+ T^*X \oplus \Lambda^- T^*X$

$\Longrightarrow \operatorname{ind} D^E = \operatorname{sign}(X)$

The index theorem of Atiyah and Singer gives a formula for these indices. In order to explain it, let R be the curvature tensor of X, considered locally as a skew-symmetric matrix of two-forms.

$$\hat{A}(X) := \det^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)$$

$$= \exp \operatorname{tr}\left(\frac{1}{2}\log\left(\frac{R/2}{\sinh(R/2)}\right)\right).$$

$$\in \Omega^{4*}(X).$$

Moreover, set

$$\mathrm{ch}(E/\Sigma) := 2^{-n/2}\,\mathrm{Str}\exp\left(F^{E/\Sigma}\right).$$

<u>Thm</u> (Atiyah–Singer). We have

$$\mathrm{ind}(D^E) = (2\pi i)^{-n/2}\int_X \hat{A}(X)\wedge \mathrm{ch}(E/\Sigma).$$

In particular, if X is spin with spinor bundle Σ, then

$$\mathrm{ind}(D^\Sigma) = \int_X \hat{A}(X).$$

<u>Thm</u> (Lichnerowicz–Schrödinger).

We have

$$(D^E)^2 = (\nabla^E)^*\nabla^E + \frac{1}{4}\mathrm{scal} + \sum_{i<k} F^{E/\Sigma}(e_i,e_j)\,c(e_i)c(e_j)$$

<u>Proof</u>. We consider only the special case $E = \Sigma$

$$(D^\Sigma)^2 \varphi = \sum_{i,j} c(e_i) \nabla_i^\Sigma c(e_j) \nabla_j^\Sigma f$$

$$= \sum_{i,j} \Big(c(e_i) c(\underbrace{\nabla_i^\Sigma e_j}_{\substack{=0 \text{ if basis is} \\ \text{synchronous at point}}}) \nabla_j^\Sigma f$$

$$+ c(e_i) c(e_j) \nabla_i^\Sigma \nabla_j^\Sigma f \Big)$$

at one point

$$= \sum_{i \neq j} c(e_i) c(e_j) \nabla_i^\Sigma \nabla_j^\Sigma f .$$

$$- \sum_{i=1}^{n} (\nabla_i^\Sigma)^2 f .$$

$$= \sum_{i<j} \underbrace{c(e_i) c(e_j) R^\Sigma(e_i, e_j) f}_{= \frac{1}{4} \sum_{i<j} \sum_{k\ell} \langle R(e_i,e_j) e_k, e_\ell \rangle c(e_k) c(e_\ell) c(e_i) c(e_j)} - (\nabla^\Sigma)^* \nabla^\Sigma f$$

Now use Bianchi-identity etc. $\square$

It turns out that

$$\int_X \hat{A}(X) \wedge ch(E/\Sigma)$$

does not depend on the metric or the geometric data on E.

Corollary. If X is spin and admits a metric of positive scalar curvature, then

$$\int_X \hat{A}(X) = 0,$$

i.e. the $\hat{A}$-genus vanishes.

This is the major obstruction to positive scalar curvature. Conversely,

$$\hat{A}(\mathbb{C}P^2) = -\frac{1}{8} \neq 0,$$

but $\mathbb{C}P^2$ has p.s.c., hence $\mathbb{C}P^2$ cannot be spin.