Aspects of Supermanifolds

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I. Motivation from Physics: 
Fermions and Bosons
Elementary particles

- Elementary particle is represented by an element $\psi$ of some Hilbert space $H$.
- System composed of multiple particles $\psi_1, \ldots, \psi_k$:

$$\psi_1 \otimes \cdots \otimes \psi_k \in H_1 \otimes \cdots \otimes H_k$$
Fermions vs. Bosons

Every elementary particle is one of the following:

1. Fermion
   - half-integer spin
   - Pauli exclusion principle:
     two fermions cannot occupy identical states
     \[ \psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_k \in \bigwedge^k H \]
   - examples: electrons, protons, neutrons, neutrinos, \ldots

2. Boson
   - integer spin
   - multiple particles can occupy identical states
     \[ \psi_1 \psi_2 \cdots \psi_k \in S^k H \]
   - examples: photons, pions, W- and Z-bosons, \ldots
Standard Model of Particle Physics

Supersymmetry as an attempt to generalise the non-relativistic standard model to a relativistic model:

- Classical SU$_6$-symmetry is insufficient (Coleman-Mandula Theorem)
- Relativistic theory might require transformations acting on particle spins (fermions ↔ bosons)
- State space for one particle:

\[ H = H_0 \oplus H_1, \]

where \( H_0 = \text{bosonic states} \) and \( H_1 = \text{fermionic states} \)

- System of \( n \) particles:

\[
\bigoplus_{k=1}^{n} \left( S^k H_0 \otimes \bigwedge^{n-k} H_1 \right)
\]
**Supersymmetries**

Supersymmetries are transformations exchanging bosonic and fermionic states.

- Infinitesimal symmetries first considered in the 1970s.
- Coordinate computations with real and Grassmann variables.
References

- V.S. Varadarajan:  
  *Supersymmetry for Mathematicians: An Introduction*  
  Chapters 1.7 and 1.8

- S. Weinberg:  
  *The Quantum Theory of Fields III*  
  Chapter 24
II. Inspired by Grothendieck:
Supermanifolds according to Berezin-Leites and Kostant
Definition of “supermanifold” based on principles of algebraic geometry.

**Example:** Affine algebraic variety $V$
- $V$ zero set of some polynomial ideal $\mathcal{I}$
- algebra of functions on $V$:
  $$\mathcal{O}(V) = \mathbb{C}[X_1, \ldots, X_n]/\mathcal{I}$$
- points $p \in V \leftrightarrow$ maximal ideals $\mathcal{M} = \mathcal{M}_p$ in $\mathcal{O}(V)$
- $\leadsto$ geometry of $V$ encoded in the functions on $V$

**Grothendieck Principle:** Generalise by
- allowing arbitrary commutative rings
- associate arbitrary prime ideals to “points”
- $\leadsto$ ringed spaces
For supermanifolds we need the following concepts:

- supercommutative rings/algebras
- sheaves
Definition: Supercommutative Algebra

An $\mathbb{R}$-algebra $\mathcal{A}$ is a \textbf{superalgebra} if it is $\mathbb{Z}_2$-graded. This means

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$$

such that $\deg(x) = \varepsilon$ if $x \in \mathcal{A}_\varepsilon$ and

$$\deg(x \cdot y) = \deg(x) + \deg(y) \mod 2$$

for all homogeneous elements $x, y \in \mathcal{A}$.

Furthermore, $\mathcal{A}$ is called \textbf{supercommutative} if

$$x \cdot y = (-1)^{\deg(x) \deg(y)} y \cdot x$$

for all homogeneous elements $x, y \in \mathcal{A}$. 
Example (the obvious one)

The exterior algebra $\mathcal{A} = \bigwedge V$ over some vector space $V$ becomes a $\mathbb{Z}_2$-graded algebra when setting

\[ \mathcal{A}_0 = \mathbb{R} \oplus \bigwedge^2 V \oplus \bigwedge^4 V \oplus \ldots \]
\[ \mathcal{A}_1 = V \oplus \bigwedge^3 V \oplus \bigwedge^5 V \oplus \ldots \]

It is supercommutative because

\[ x \wedge y = -y \wedge x \]

for all $x, y \in V$.

(This leads to the irritating funny fact that the alternating algebra in the category of $\mathbb{R}$-algebras becomes the symmetric algebra in the category of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras.)
Definition: Sheaf

Let $X$ be a topological space.

A sheaf $\mathcal{O}$ of (super)commutative rings is a collection of maps

$$\{ U \mapsto \mathcal{O}(U) \}_{U \text{ open in } X},$$

such that $\mathcal{O}(U)$ is a (super)commutative ring satisfying:

1. For all $W \subset U \subset V$ there exist restriction homomorphisms

   $$\varrho_{U}^{V} : \mathcal{O}(U) \to \mathcal{O}(W)$$

   satisfying $\varrho_{W}^{U} \circ \varrho_{V}^{U} = \varrho_{V}^{W}$ and $\varrho_{U}^{U} = \text{id}_{U}$.

2. If $U = U_1 \cup \ldots \cup U_k$ and $f_i \in \mathcal{O}(U_i)$, then

   $$\exists f \in \mathcal{O}(U) \forall i : \varrho_{U_i}^{U}(f) = f_i \iff \varrho_{U_i \cap U_j}^{U}(f_i) = \varrho_{U_i \cap U_j}^{U}(f_j)$$

   and $f$ is unique whenever it exists.

$(X, \mathcal{O})$ is called a (super)ringed space.
Sheaves generalise . . .

- the algebras of $C^\infty$-functions on the open subsets of a smooth manifold;
- the algebras of regular functions on open subsets of an affine algebraic variety.

Common abuse of language: $\mathcal{O}(U)$ are the “functions on $U$”.
Definition: Morphisms of Ringed Spaces

Let \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) be (super)ringed spaces.

A morphism \((\psi, \psi^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) consists of

1. a continuous map \(\psi : X \to Y\)
2. and a collection \(\psi^*\) of “pullback” homomorphisms

\[ \{ \psi_W^* : \mathcal{O}(W) \to \mathcal{O}(\psi^{-1}(W)) \} \text{ open in } Y \]

which commute with the restriction map.
They must preserve the \(\mathbb{Z}_2\)-grading in the super category.

Define isomorphisms accordingly.
Remark

If the rings in the sheaves are rings of functions, then the $\psi^*$ are the pullback maps for these functions from $\mathcal{W}$ to $\psi^{-1}(\mathcal{W})$. 
Example

A superdomain $U^n|^{k}$ consists of an open subset $U \subseteq \mathbb{R}^n$ and the sheaf given by

$$\mathcal{O}(W) = C^\infty(W) \otimes \bigwedge \mathbb{R}^k, \quad W \text{ open in } U.$$  

Notation:

$$\mathcal{O}(W) = C^\infty(W)[\vartheta_1, \ldots, \vartheta_k]$$

$$= C^\infty(x_1, \ldots, x_n)[\vartheta_1, \ldots, \vartheta_k],$$

where $x_1, \ldots, x_n$ are coordinates on $U$ and $\vartheta_1, \ldots, \vartheta_k$ are generators of the exterior algebra.
Definition: Supermanifold

Let $|M|$ be a topological space.

A superringed space $M = (|M|, \emptyset)$ is a supermanifold of superdimension $n|k\rangle$, if there exists a cover of $|M|$ by open sets $W$ such that

$$(W, \emptyset_{|W}) \cong U_{W}^{n|k} \quad (U_{W}^{n|k}$ some superdomain).$$

In other words: $M$ is locally isomorphic to $\mathbb{R}^{n|k}$. 
Some Properties of Supermanifolds (I)

Given $M$, the space $|M|$ becomes a (classical) smooth manifold

$$M_{\text{red}} = (|M|, \mathcal{O}/\mathcal{J}),$$

where

$$\mathcal{J} = \mathcal{O}_1 + \mathcal{O}_1^2$$

is the sheaf of ideals generated by the odd elements.
Some Properties of Supermanifolds (II)

Associate to $f \in \mathcal{O}(U)$ its value $f(x)$ at $x \in U$:

- The unique $\lambda \in \mathbb{R}$ such that $f - \lambda$ is not invertible in any neighbourhood of $x$.
- !!! This does not turn $f$ into a classical function on $U$:

$$\forall x \in U : f_1(x) = f_2(x) \nRightarrow f_1 = f_2$$

In particular, any $f \in \mathcal{O}(U)_1$ has constant value 0.

The stalk $\mathcal{O}_x$ at $x \in |M|$ is a local ring with maximal ideal $\mathcal{M}_x = \ker(f \mapsto f(x))$.

Consequence:
A morphism $(\varphi, \varphi^*) : M \rightarrow N$ of supermanifolds induces morphisms $\varphi_x^* : \mathcal{O}_{\varphi(x)} \rightarrow \mathcal{O}_x$ mapping $\mathcal{M}_{\varphi(x)}$ to $\mathcal{M}_x$. 
Morphisms in Coordinates

Let $U^{p|q}$ be a superdomain with coordinates $x_i, \vartheta_j$, and let $M$ be a supermanifold.

Theorem:

- Let $(\psi, \psi^*) : M \to U^{p|q}$ be a morphism. If

  \[ a_i = \psi^*(x_i) \quad (i = 1, \ldots, p), \quad b_i = \psi^*(\vartheta_i) \quad (i = 1, \ldots, q), \]

  then $a_i \in \mathcal{O}(M)_0$ and $b_i \in \mathcal{O}(M)_1$.

- Conversely, if $a_i \in \mathcal{O}(M)_0$ and $b_i \in \mathcal{O}(M)_1$ are given, then there exists a unique morphism $(\psi, \psi^*) : M \to U^{p|q}$ such that

  \[ a_i = \psi^*(x_i) \quad (i = 1, \ldots, p), \quad b_i = \psi^*(\vartheta_i) \quad (i = 1, \ldots, q). \]
Morphisms in Coordinates

Given $a_i \in \mathcal{O}(M)_0$ and $b_i \in \mathcal{O}(M)_1$, there exists a unique morphism $(\psi, \psi^*) : M \to U^{p|q}$ such that

$$a_i = \psi^*(x_i) \quad (i = 1, \ldots, p), \quad b_i = \psi^*(\vartheta_i) \quad (i = 1, \ldots, q).$$

Sketch of Proof:

- **Existence:**
  - $\vartheta_1, \ldots, \vartheta_q$ algebraically generate the exterior algebra, so $\psi^*(\vartheta_i)$ can be chosen arbitrarily in $\mathcal{O}(M)_1$.
  - $\Rightarrow$ enough to construct homomorphism $C^\infty(U) \to \mathcal{O}(M)_0$ with $x_i \mapsto a_i$
  - $a_i = y_i + \xi_i$, where $\xi_i \in \mathcal{I}$ is nilpotent and $y_i$ is not
  - for $f \in C^\infty(U)$ define by formal Taylor expansion:

$$\psi^*(f) = f(y_1 + \xi_1, \ldots, y_p + \xi_p) = \sum_{k} \frac{1}{k!} (\partial^k f)(y_1, \ldots, y_p) \cdot \xi^k$$

  this sum is finite because the $\xi_i$ are nilpotent

- **Uniqueness:** Use approximation of smooth functions by polynomial functions.
Example

Define morphism \((\psi, \psi^*) : \mathbb{R}^{1|2} \rightarrow \mathbb{R}^{1|2}\) by

\[ \psi(x) = x \]

and

\[ \psi^*(x) = x + \vartheta_1 \vartheta_2, \]
\[ \psi^*(\vartheta_i) = \vartheta_i. \]

For arbitrary \(f \in C^\infty(\mathbb{R}^1)\), we then have

\[ \psi^*(f) = f(x) + f'(x) \vartheta_1 \vartheta_2. \]
**Geometric Intuition?**

- “*M is essentially a classical manifold surrounded by a cloud of odd stuff.*” (V.S. Varadarajan)
- “*M can be thought of as M_{red}, surrounded by a nilpotent fuzz.*” (P. Deligne)

But see Varadarajan, Chapter 4.5, for the notion of points provided by the *functor of points.*
References

- P. Deligne et al.:  
  *Quantum Fields and Strings: A Course for Mathematicians I*  
  Part 1, Chapter 2

- Yu.I. Manin:  
  *Gauge Theory and Complex Geometry*  
  Chapter 4

- V.S. Varadarajan:  
  *Supersymmetry for Mathematicians: An Introduction*  
  Chapter 4
III. Examples
examples are hard to find...
If $E$ is a real vector bundle over the differentiable manifold $M_0$, let $\bigwedge E$ denote the associated exterior bundle. Then $M_E = (M_0, \Gamma(\bigwedge E))$ is a supermanifold.

**Batchelor’s Theorem:**
Every supermanifold over $M_0$ is (non-canonically) isomorphic to $M_E$ for some vector bundle $E$ over $M_0$.

**Remark:**
- Many more morphisms in the super category than in the differentiable category.
- Batchelor’s Theorem does not hold for analytic supermanifolds.
Complex Projective Superspace

Let $\mathbb{P}^n$ be the complex projective $n$-space.

Define the complex projective superspace $\mathbb{P}^n|k$ as follows:

- For $V$ open in $\mathbb{P}^n$, let $V'$ denote its preimage in $\mathbb{C}^{n+1} \setminus \{0\}$.
- Define action of $t \in \mathbb{C}^\times$ on $\mathcal{A}(V') = \mathcal{H}(V')[\vartheta_1, \ldots, \vartheta_q]$ by
  \[
  t \cdot \sum_i f_i(z) \vartheta^i = \sum_i t^{-|i|} f_i(t^{-1}z) \vartheta^i
  \]
  and set $\mathcal{O}(V) = \mathcal{A}(V')^{\mathbb{C}^\times}$.
- Then $\mathcal{O}(V) \cong \mathcal{H}(X)[\vartheta_1, \ldots, \vartheta_q]$ for some affine subspace $X \subset \mathbb{C}^{n+1}$.
- $\mathcal{O}$ is a sheaf of supercommuting $\mathbb{C}$-algebras on $X$ and
  \[
  \mathbb{P}^n|k = (\mathbb{P}^n, \mathcal{O}).
  \]

See also Manin, Chapter 4.3.
Lie Supergroups

A Lie Supergroup $G$ is a supermanifold with a morphism

$$\mu : G \times G \rightarrow G$$

such that there exists a unit $e : \mathbb{R}^{0|0} \rightarrow G$ and an inverse map $\iota : G \rightarrow G$ such that standard diagrams commute.

The linear supergroup $\text{GL}(p|q)$ is an open supersubmanifold of $\mathbb{R}^{p^2+q^2|2pq}$ given as follows:

- Write coordinates in $\mathbb{R}^{p^2+q^2|2pq}$ as follows:

$$\begin{pmatrix} A_0 & B_1 \\ C_1 & D_0 \end{pmatrix},$$

where the matrices $A_0, D_0$ contain the even coordinates and $B_1, C_1$ the odd coordinates.

- $\text{GL}(p|q)$ is the open supersubmanifold defined by $\det(A_0) \det(D_0) \neq 0$. 
References

- M. Batchelor:
  *The Structure of Supermanifolds*
  Trans. Amer. Math. Soc. 253, 1979

- Yu.I. Manin:
  *Gauge Theory and Complex Geometry*
  Chapter 4.3

- V.S. Varadarajan:
  *Supersymmetry for Mathematicians: An Introduction*
  Chapter 4
IV. Differential Calculus and Berezin-Integration
Definition: Derivation

A derivation $D$ on an $\mathbb{R}$-superalgebra $\mathcal{A}$ is an $\mathbb{R}$-linear map satisfying

$$D(x \cdot y) = D(x) \cdot y + (-1)^{\deg(x) \deg(D)} x \cdot D(y).$$
Definition: Vector Fields

Let $M$ be a supermanifold. A vector field on $M$ is a derivation of $\mathcal{O}$, that is a family of derivations

$$\{ D_U : \mathcal{O}(U) \to \mathcal{O}(U) \}_{U \text{ open in } |M|}.$$ 

The derivations of $M$ form the tangent sheaf $\mathcal{T}M$.

In coordinates:

- Extend derivations $D \in C^\infty(U)$ via $D(\vartheta_i) = 0$.
- Define $\partial_{\vartheta_i}$ by $\partial_{\vartheta_i}(\vartheta_k) = \delta_{ik}$.
- The derivations
  $$\partial_{x_j}, \partial_{\vartheta_i}$$

  form a module basis for the derivations of $\mathcal{O}(U)$. 
Definition: Tangent Vectors and the Differential

Let $M$ be a supermanifold and $p \in |M|$. A tangent vector at $p$ is a derivation $v : \mathcal{O}_p \to \mathbb{R}$ of the stalk $\mathcal{O}_p$ into $\mathbb{R}$. They form the tangent space $T_pM$ at $p$.

The differential $d\psi_p$ of a morphism $(\psi, \psi^*) : M \to N$ of supermanifolds is the morphism

$$d\psi_p : T_pM \to T_{\psi(p)}N, \quad v \mapsto v \circ \psi_p^*.$$

Vector fields and differentials have properties similar to the classical case (see Varadarajan, Chapter 4.4).
The integral on $\mathbb{R}[\vartheta_1, \ldots, \vartheta_k]$ is defined by

\[
\int \vartheta^i = 0, \quad \text{if } |i| < k, \\
\int \vartheta_1 \cdots \vartheta_k = 1.
\]

Observe that

\[
\int = \partial_{\vartheta_k} \cdots \partial_{\vartheta_1}.
\]
Definition: Berezin Integral

On a superdomain $U^{n|k}$ the Berezin integral for compactly supported sections $s = \sum_i s_i \vartheta^i$ is defined as

$$\int : \mathcal{O}_c(U) \to \mathbb{R}, \quad s \mapsto \int_U s_{(1,\ldots,k)}(x) \, d^k x.$$
Change of Variables

For a change of variables given locally as

$$\psi(x, \vartheta) = (y, \eta),$$

let

$$J_\psi = \begin{pmatrix} \frac{\partial y}{\partial x} & -\frac{\partial y}{\partial \vartheta} \\ \frac{\partial y}{\partial \eta} & \frac{\partial \eta}{\partial \vartheta} \end{pmatrix}.$$  

**Theorem:**

$$\int s = \int \psi^*(s) \text{Ber}(J_\psi)$$
References

V.S. Varadarajan:
*Supersymmetry for Mathematicians: An Introduction*
Chapters 4.4 and 4.6